International Journal of Scientific Research and Reviews

Study of Inter Relation Between Two Types of Continuous Functions On Convex Topological Space

Bijoy Samanta

Assistant Professor in Mathematics

Department of Mathematics, Shibpur Dinobundhoo Institution (College) 412/1, G.T. Road (S),
Shibpur , Howrah – 711 102 , W.B., India E-mail : bijoysamanta@yahoo.com

ABSTRACT

The concept of $\delta - \mathcal{C}$ continuity and $\delta_\star - \mathcal{C}$ continuity already have been introduced earlier on convex topological space $(X , \tau , \mathcal{C})$ where τ is the topology and $\mathcal{C}$ is the convexity on the same underlying set X . In this paper I have mainly investigated the inter relation between these two types of continuity .

AMS SUBJECT CLASSIFICATION : 52A01 , 54C10 .

KEYWORDS : Convex topological space , $\delta - \mathcal{C}$ continuous function , $\delta_\star - \mathcal{C}$ continuous function .

*Corresponding author:

Dr. Bijoy Samanta

Assistant Professor in Mathematics

Department of Mathematics , Shibpur Dinobundhoo Institution (College)

412/1 , G.T. Road (S) , Shibpur , Howrah – 711 102 , W.B. , India

E-mail : bijoysamanta@yahoo.com

1. INTRODUCTION

The development of "abstract convexity" has emanated from different sources in different ways ; the first type of development basically banked on generalization of particular problems such as separation of convex *sets* [1], *extremality* [2,3] or continuous *selection* [4] . The second type of development lay before the reader such axiomatizations , which in every case of design , express particular point of view of convexity . With the view point of generalized topology which enters into convexity via the closure or hull operator , Schmidt and Hammer, , introduced some axioms to explain abstract convexity . The arising of convexity from algebraic operations and the related property of domain finiteness receive attentions in Birch off and Frink , Schmidt, Hammer .

The axiomatizations as proposed by M.L.J. Van De Vel in his paper Theory of Convex *Structure* [5] will be followed through out in this paper .

The author has discussed in " Topology and Convexity on the same *set* [6] " and introduced the compatibility of the topology with a convexity on the same underlying set . At the very early stage of this paper we have set aside this concept of compatibility and started just with a triplet $(X , \tau , \mathcal{C})$ and call it convex topological space only to bring back "compatibility" in another way subsequently . With this compatibility , Van De Vel has called the triplet $(X , \tau , \mathcal{C})$ a topological convex structure

In this paper , Art. 2 deals with some early definitions , results and in Art. 3 I have discussed mainly inter relation between $\delta - \mathcal{C}$ continuous function and $\delta_\star - \mathcal{C}$ continuous function .

2. PREREQUISITES :

Definition 2.1 [6] **:** Let X be a non empty set . A family $\mathcal{C}$ of subsets of the set X is called a convexity on X if

1. $\phi , X \in \mathcal{C}$
2. $\mathcal{C}$ is stable for intersection , i.e. if $\mathcal{D} \subseteq \mathcal{C}$ is non empty , then $\cap \mathcal{D} \in \mathcal{C}$
3. $\mathcal{C}$ is stable for nested unions , i.e. if $\mathcal{D} \subseteq \mathcal{C}$ is non empty and totally ordered by set inclusion, then $\cup \mathcal{D} \in \mathcal{C}$.

The pair $(X , \mathcal{C})$ is called a convex structure . The members of $\mathcal{C}$ are called convex sets and their complements are called concave sets .

Definition 2.2 [6] **:** Let $\mathcal{C}$ be a convexity on set X . Let $A \subseteq X$. The convex hull of A is denoted by $co(A)$ and defined by $co(A) = \cap \{ C : A \subseteq C \in \mathcal{C} \}$.

Note 2.3 [6] **:** Let $(X , \mathcal{C})$ be a convex structure and let Y be a subset of X . The family of sets $\mathcal{C}_Y = \{ C \cap Y : C \in \mathcal{C} \}$ is a convexity on Y; called the relative convexity of Y.

Note 2.4 [6] **:** The hull operator co_Y of a subspace $(Y , \mathcal{C}_Y)$ satisfy the following :

$$\forall A \subseteq Y : \ co_Y(A) = co(A) \cap Y .$$

Definition 2.5 [6]**:** Let $(X , \mathcal{C})$ be a convex structure and let τ be a topology on X. Then τ is said to be compatible with the convex structure $(X , \mathcal{C})$ if all polytopes of $\mathcal{C}$ are closed in τ where polytopes means convex hull of a finite set. Also the triplet $(X , \tau , \mathcal{C})$ is then called topological convex structure.

Note 2.6 [6] **:** Let $(X , \tau , \mathcal{C})$ be a topological convex structure. Then collection of all closed sets in (X , τ) are subset of $\mathcal{C}$.

Definition 2.7 [7] **:** Let (X , τ) be a topological space and let $\mathcal{C}$ be a convexity on X. Then the triplet $(X , \tau , \mathcal{C})$ is called a convex topological space (CTS in short).

Theorem 2.8 [7] **:** Let $(X , \tau , \mathcal{C})$ be a convex topological space. Let A be a subset of X. Consider the set A_*, where A_* is defined as follows : $A_* = \{x \in X : co(U) \cap A \neq \emptyset, x \in U \in \tau\}$. Then the collection $\tau_* = \{A^c : A \subseteq X , A = A_*\}$ is a topology on X such that $\tau_* \subseteq \tau$.

Definition 2.9 [8] **:** Let $(X , \tau , \mathcal{C}_1)$ and $(Y , \sigma , \mathcal{C}_2)$ be two convex topological spaces. A function $f : (X , \tau , \mathcal{C}_1) \to (Y , \sigma , \mathcal{C}_2)$ is said to be $\delta - \mathcal{C}$ continuous if for each $x \in X$ and each open nbd. V of $f(x)$, there exists an open nbd. U of x such that $f\big(int(U_*)\big) \subseteq int(V_*)$.

Definition 2.10 [9] **:** Let $(X , \tau , \mathcal{C}_1)$ and $(Y , \sigma , \mathcal{C}_2)$ be two convex topological spaces. A function $f : (X , \tau , \mathcal{C}_1) \to (Y , \sigma , \mathcal{C}_2)$ is said to be $\delta_\star - \mathcal{C}$ continuous if for each $x \in X$ and each open nbd. V of $f(x)$, there exists an open nbd. U of x such that $f\big(int(co(U))\big) \subseteq int\big(co(V)\big)$.

Definition 2.11 [10] **:** A convex topological space $(X , \tau , \mathcal{C})$ is said to be an $SC - R$ space if for each $x \in X$ and each open nbd. V of x there exists an open nbd. U of x such that $x \in U \subseteq int(U_*) \subseteq V$.

Definition 2.12 [11] **:** A convex topological space $(X , \tau , \mathcal{C})$ is said to be a semi $\mathcal{C}$- regular space if for each $x \in X$ and each open nbd. V of x there exists an open nbd. U of x such that $x \in U \subseteq int\big(co(U)\big) \subseteq V$.

3. COMPARISON BETWEEN $\delta - \mathcal{C}$ CONTINUOUS AND $\delta_\star - \mathcal{C}$ CONTINUOUS FUNCTIONS :

Already I have discussed detail the concept of $\delta - \mathcal{C}$ continuity[8] and $\delta_\star - \mathcal{C}$ continuity[9] on convex topological space. Now I will show that these two concepts are independent in general which follow from the next two examples.

Example 3.1 : Let us consider the function $f : (X, \tau, C_1) \rightarrow (X, \sigma, C_2)$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $C_1 = \{\emptyset, X\}$, $\sigma = \{\emptyset, X, \{a\}\}$, $C_2 = \{\emptyset, X, \{a, b\}\}$ and f is the identity mapping I_X on X.

In the convex topological space (X, σ, C_2), we see that $\{a\}_\star = \{b\}_\star = \{c\}_\star = X$. This shows that the function f is $\delta - C$ continuous.

Again for the point a in (X, τ, C_1) we consider the open nbd. $V = \{a\}$ of $f(a) = a$ in (X, σ, C_2). In the CTS (X, σ, C_2), $co(\{a\}) = \{a, b\}$ and $int(co(\{a\})) = \{a\}$. Also in the CTS (X, τ, C_1), $(\{a\}) = co(\{a, b\}) = co(X) = X$. Thus there is no open nbd. $U \in \tau$ of a such that $f\left(int(co(U)) \right) \subseteq \left(int(co(V)) \right)$. This shows that f is not $\delta_\star - C$ continuous.

Hence we conclude that $\delta - C$ continuity $\nRightarrow \delta_\star - C$ continuity.

Example 3.2 : Let us consider the function $f : (X, \tau, C_1) \rightarrow (X, \sigma, C_2)$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b\}\}$, $C_1 = \{\emptyset, X, \{b\}\}$, $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $C_2 = \{\emptyset, X, \{b\}\}$ and f is the identity mapping I_X on X.

In the CTS (X, τ, C_1), non-empty open sets are $\{b\}$, X and $int(co(\{b\})) = \{b\}$, $(co(X)) = X$. Also in the CTS (X, σ, C_2), non-empty open sets are $\{a\}$, $\{b\}$, $\{a, b\}$, X and $int(co(\{a\})) = int(co(\{a, b\})) = int(co(X)) = X$, $int(co(\{b\})) = \{b\}$. So we see that for each $x \in X$ in (X, τ, C_1) and each open nbd. V of $f(x)$ in (X, σ, C_2), there exists an open nbd. U of x such that $f\left(int(co(U)) \right) \subseteq int(co(V))$. This shows that the function f is $\delta_\star - C$ continuous.

Again for the point a in (X, τ, C_1) we consider the open nbd. $V = \{a\}$ of $f(a) = a$ in (X, σ, C_2). Now in the CTS (X, σ, C_2), $(\{a\}_\star) = int(\{a, c\}) = \{a\}$. Also in the CTS (X, τ, C_1), open nbd. of a is X and $int(X_\star) = X$. Thus there is no open nbd. $U \in \tau$ of a such that $f\left(int(U_\star) \right) \subseteq int(V_\star)$. This shows that f is not $\delta - C$ continuous.

Hence we conclude that $\delta_\star - C$ continuity $\nRightarrow \delta - C$ continuity.

Now I will discuss under what conditions these two concepts coincide.

Theorem 3.3 : If a function $f : (X, \tau, C_1) \rightarrow (Y, \sigma, C_2)$ be $\delta - C$ continuous and Y is an $SC - R$ space where τ is compatible with C_1, then f is $\delta_\star - C$ continuous function.

Proof : Since τ is compatible with C_1, all closed sets of X are in C_1. Then for any $P \subseteq X$, $co(P) \subseteq \bar{P} \subseteq P_\star$ and so $int(co(P)) \subseteq int(P_\star)$. Let $x \in X$ and V be any open nbd. of $f(x)$. Since Y is an $SC - R$ space, there exists an open set W such that $f(x) \in W \subseteq int(W_\star) \subseteq V$.

Again f is $\delta - \mathcal{C}$ continuous . So there exists an open nbd. U of x such that $f\left(int(U_\star)\right) \subseteq int(W_\star)$. Now $int(W_\star) \subseteq V \subseteq co(V) \Rightarrow int(W_\star) \subseteq int\left(co(V)\right)$. This shows that $f\left(int(co(U))\right) \subseteq f\left(int(U_\star)\right) \subseteq int(W_\star) \subseteq int\left(co(V)\right)$. Hence f is $\delta_\star - \mathcal{C}$ continuous function .

Theorem 3.4 : If a function $f : (X , \tau , \mathcal{C}_1) \rightarrow (Y , \sigma , \mathcal{C}_2)$ be $\delta_\star - \mathcal{C}$ continuous and X is an $S\mathcal{C} - R$ space where σ is compatible with $\mathcal{C}_2$, then f is $\delta - \mathcal{C}$ continuous function .

Proof : Given σ is compatible with $\mathcal{C}_2$. Then for any $P \subseteq Y$, $co(P) \subseteq \bar{P} \subseteq P_\star$ and so $int\left(co(P)\right) \subseteq int(P_\star)$. Let $x \in X$ and V be any open nbd. of $f(x)$. Since f is $\delta_\star - \mathcal{C}$ continuous , there exists an open nbd. U of x such that $f\left(int(co(U))\right) \subseteq int(co(V))$. So $f\left(int(co(U))\right) \subseteq int(co(V)) \subseteq int(V_\star)$. Again X is an $S\mathcal{C} - R$ space . So there exists an open set W such that $x \in W \subseteq int(W_\star) \subseteq U$. Now $int(W_\star) \subseteq U \subseteq co(U) \Rightarrow int(W_\star) \subseteq int(co(U))$. This shows that $f(int(W_\star)) \subseteq f\left(int(co(U))\right) \subseteq int(V_\star)$. Hence f is $\delta - \mathcal{C}$ continuous function .

Theorem 3.5 : If a function $f : (X , \tau , \mathcal{C}_1) \rightarrow (Y , \sigma , \mathcal{C}_2)$ be $\delta - \mathcal{C}$ continuous where X is a semi $\mathcal{C}$- regular space and Y is an $S\mathcal{C} - R$ space , then f is $\delta_\star - \mathcal{C}$ continuous function .

Proof : Let $x \in X$ and V be any open nbd. of $f(x)$. Since Y is an $S\mathcal{C} - R$ space , there exists an open set W of $f(x)$ such that $f(x) \in W \subseteq int(W_\star) \subseteq V$. So $int(W_\star) \subseteq co(V) \subseteq int(co(V))$. Again f is $\delta - \mathcal{C}$ continuous . So there exists an open nbd. Z of x such that $f(int(Z_\star)) \subseteq int(W_\star)$. Also X is a semi $\mathcal{C}$- regular space . Thus there exists an open set U such that $x \in U \subseteq int(co(U)) \subseteq int(Z_\star)$. So $f\left(int(co(U))\right) \subseteq f(int(Z_\star)) \subseteq int(W_\star) \subseteq int(co(V))$. This shows that f is $\delta_\star - \mathcal{C}$ continuous function .

Theorem 3.6 : If a function $f : (X , \tau , \mathcal{C}_1) \rightarrow (Y , \sigma , \mathcal{C}_2)$ be $\delta_\star - \mathcal{C}$ continuous where X is an $S\mathcal{C} - R$ space and Y is a semi $\mathcal{C}$- regular space , then f is $\delta - \mathcal{C}$ continuous function .

Proof : Let $x \in X$ and V be any open nbd. of $f(x)$. Now $int(V_\star)$ is an open nbd. of $f(x)$ and Y is an semi $\mathcal{C}$- regular space .So there exists an open set W of $f(x)$ such that $f(x) \in W \subseteq int(co(W)) \subseteq int(V_\star)$. Again f is $\delta_\star - \mathcal{C}$ continuous . So there exists an open nbd. Z of x such that $f\left(int(co(Z))\right) \subseteq int(co(W))$. Also X is a $S\mathcal{C} - R$ space . Thus there exists an open set U such that $x \in U \subseteq int(U_\star) \subseteq int(co(Z))$. So $f(int(U_\star)) \subseteq f\left(int(co(Z))\right) \subseteq int(co(W)) \subseteq int(V_\star)$. This shows that f is $\delta - \mathcal{C}$ continuous function .

Theorem 3.7 : If a function $f : (X , \tau , C_1) \rightarrow (Y , \sigma , C_2)$ be continuous and X is an $SC - R$ space , then f is $\delta - C$ continuous function .

Proof : Let $x \in X$ and V be any open nbd. of $f(x)$. Now $int(V_\star)$ is an open nbd. of $f(x)$. Since f is continuous , there exists an open nbd. W of x such that $f(W) \subseteq int(V_\star)$. Again X is an $SC - R$ space . So there exists an open set $U \in \tau$ such that $\in U \subseteq int(U_\star) \subseteq W$. Thus $f\big(int(U_\star)\big) \subseteq f(W) \subseteq int(V_\star)$. This shows that f is $\delta - C$ continuous function .

Theorem 3.8 : If a function $f : (X , \tau , C_1) \rightarrow (Y , \sigma , C_2)$ be continuous and X is a semi C- regular space , then f is $\delta_\star - C$ continuous function .

Proof : Let $x \in X$ and V be any open nbd. of $f(x)$. Now $int\big(co(V)\big)$ is an open nbd. of $f(x)$. Since f is continuous , there exists an open nbd. W of x such that $f(W) \subseteq int\big(co(V)\big)$. Again X is a semi C- regular space . So there exists an open set $U \in \tau$ such that $x \in U \subseteq int\big(co(U)\big) \subseteq W$. Thus $f\left(int\big(co(U)\big)\right) \subseteq f(W) \subseteq int\big(co(V)\big)$. This shows that f is $\delta_\star - C$ continuous function .

Theorem 3.9[10] : If a function $f : (X , \tau , C_1) \rightarrow (Y , \sigma , C_2)$ be $\delta - C$ continuous and Y is an $SC - R$ space , then f is continuous function .

Theorem 3.10[11] : If a function $f : (X , \tau , C_1) \rightarrow (Y , \sigma , C_2)$ be $\delta_\star - C$ continuous and Y is a semi C- regular space , then f is continuous function .

Theorem 3.11 : Let $f : (X , \tau , C_1) \rightarrow (Y , \sigma , C_2)$ be a function where X , Y are $SC - R$ spaces . Then f is continuous iff it is $\delta - C$ continuous .

Proof : Follows from the Theorems 3.7 and 3.9 .

Theorem 3.12 : Let $f : (X , \tau , C_1) \rightarrow (Y , \sigma , C_2)$ be a function where X , Y are semi C- regular spaces . Then f is continuous iff it is $\delta_\star - C$ continuous .

Proof : Follows from the Theorems 3.8 and 3.10 .

Theorem 3.13 : Let $f : (X , \tau , C_1) \rightarrow (Y , \sigma , C_2)$ be a function where X , Y are $SC - R$ and semi C- regular spaces . Then f is $\delta - C$ continuous iff it is $\delta_\star - C$ continuous .

Proof: Follows from the Theorems 3.11 and 3.12 .

REFERENCES :

1. Ellis J. W. ; A general set separation theorem , Duke Math. J. , 1952; $19 : 417 - 421$.

2. Fan K. ; On the Krein – Milman Theorem , Proc. Symp. Pure Math. , 1963 ; 7 : 211 - 221 .

3. Davies E. B. ; A generalized theory of convexity , Proc. London Math. Soc. , 1967 ; 17 :644 - 652 .

4. Michael E. ; Convex structure and continuous selection , Canadian J. Math . , 1959 ; 11: 556 – 575 .

5. Van De Vel M.; A selection theorem for topological convex structures , Trans. Amer. Math. Soc. , 1993 ; 336 (2): 463 – 496 .

6. Van De Vel M. L. J.; Theory of convex structures , North Holland , 1993 .

7. Ganguly S. and Samanta Bijoy; Some functions related to convex topological spaces , Rev. Acad. Canar. Cienc. , 2006 ; 18: 9 – 21 .

8. Samanta Bijoy; A continuous function on Convex topological Space , International Journal of Research and Analytical Reviews , 2018 ; 5 (3) , 44a – 47a .

9. Samanta Bijoy ; Convex Topological Space and Delta Star-C-continuous function , , Communicated .

10. Samanta Bijoy ; Convex topological Space and some continuous functions , International Journal of Research and Analytical Reviews , 2018 ; 5 (4): 76 – 79

11. Samanta Bijoy ; Investigation of inter relation among different types of continuous functions on convex topological space , International Journal of Scientific Research and Reviews , 2019 ; 8(1): 2921 –2928 .

12. Ganguly S. and Samanta Bijoy; $\theta - \mathcal{C}$ somewhat continuous function , Rev. Acad. Canar. Cienc. , 2008 ; 20: 63– 73 .

13. Samanta Bijoy ; Some open sets in Convex topological Space , International Journal of Research and Analytical Reviews , 2018 ; 5(3): 493z – 495z .

14. Birkhoff   G. and Frink O. ; Represents of lattices by sets , Trans. Amer. Math. Soc. , 1948 ; 64: 229 - 315 .

15. Hammer P. C.; General topology , symmetry and convexity , Trans . Wisc. Acad. Sci. Arts Lett. 1955; 44: 221 – 255 .

16. Hammer P. C. ; Extented topology ; domain finiteness , Indag. Math. , 1963 ; 25: 200 – 212 .

17. Hammer P. C. ; Semi spaces and the topology from convexity , Proc. Symp. Pure Math . , 1963 ; 7: 305 – 316 .

18. Schmidt J. ; Einige grundlegendle Begriffe und Satze aus der Theorie der Hullenoperatoren , Bericht Uber die Mathematiker – Tagung in Berlin , 1953; 21 – 48 .