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A brief review over the logistic and theta logistic growth model : Deterministic and stochastic approach

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ABSTRACT

Growth curve model plays a major role in describing the biological phenomena of any species. In ecology, relative growth rate (henceforth, RGR) can be treated as proxy of species fitness. Logistic growth model is the simplest and the first member of the density dependent family. This model described the linear size – RGR relationship, which is not general for most of the species. This non linear relationship (concave & convex both) can be captured through the theta – logistic model, where θ is the curvature parameter. Bifurcation analysis has been done with respect to this curvature parameter. Moreover, the stochastic version of theta – logistic model is also present in this article.

KEYWORDS: Logistic model, Theta - logistic model, Bifurcation, Stochasticity

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1 INTRODUCTION

Growth curve can be broadly classified into two categories, viz., density independent and and dependent. The simplest law of population growth is characterized by the constant relative growth rate (henceforth, RGR) leading to an exponential function, popularly known as Malthusian law (Malthus, 1798). This is the first member of density independent family. The density dependent models refer to the population dynamics with growth rate depending on some functions of population size or density.

$$\frac{1}{N(t)}\frac{dN(t)}{dt} = r \tag{1}$$

This law (1) indicates the species natural propensity towards any ecosystem. Therefore, if nature will not give any resistance to the species then their growth will be unbounded. Hence, this unlimited growth has a close synergy with the Newton's first law of motion in particle dynamics.

2 MATERIAL AND METHODS

2.1 Logistic Growth Curve Model

In reality, nature will not allow any species to grow unboundedly. Due to the intra-species competition, limited food resource species growth will be declined but at a large time interval population size of the cohort will be constant. Observing this phenomena, Verhulst gave a model which is modified by Pearl latter, termed as Pearl - Verhulst model[1]. This model is popularly known as Logistic Model.

$$\frac{1}{N(t)}\frac{dN(t)}{dt} = r\left(1 - \frac{N(t)}{K}\right) \tag{2}$$

Here, N(t) represents the population size of any cohort at any instant time t. Moreover r, K being the model parameters describing the intrinsic growth rate (henceforth, IGR) and carrying capacity i.e. the maximum population size of any cohort respectively. Moreover, the left hand side of equation (2) denotes the RGR. The term $\left(1 - \frac{N(t)}{K}\right)$ is responsible for the nature's negative feedback.

2.1.1 *Relationship of Rgr With Population Size*

Now, denoting $\frac{1}{N(t)}\frac{dN(t)}{dt}$ as $R(t)$ equation (2) can be written as

$$R(t) = r\left(1 - \frac{N(t)}{K}\right)$$

$$or,\ R(t) = r - \frac{rN(t)}{K}$$

$$or,\ R(t) + \frac{rN(t)}{K} = r$$

$$or, \frac{R(t)}{r} + \frac{N(t)}{K} = 1$$

This represents the intercepted form of a straight line between R(t) and N(t), which is depicted in the Figure (1).

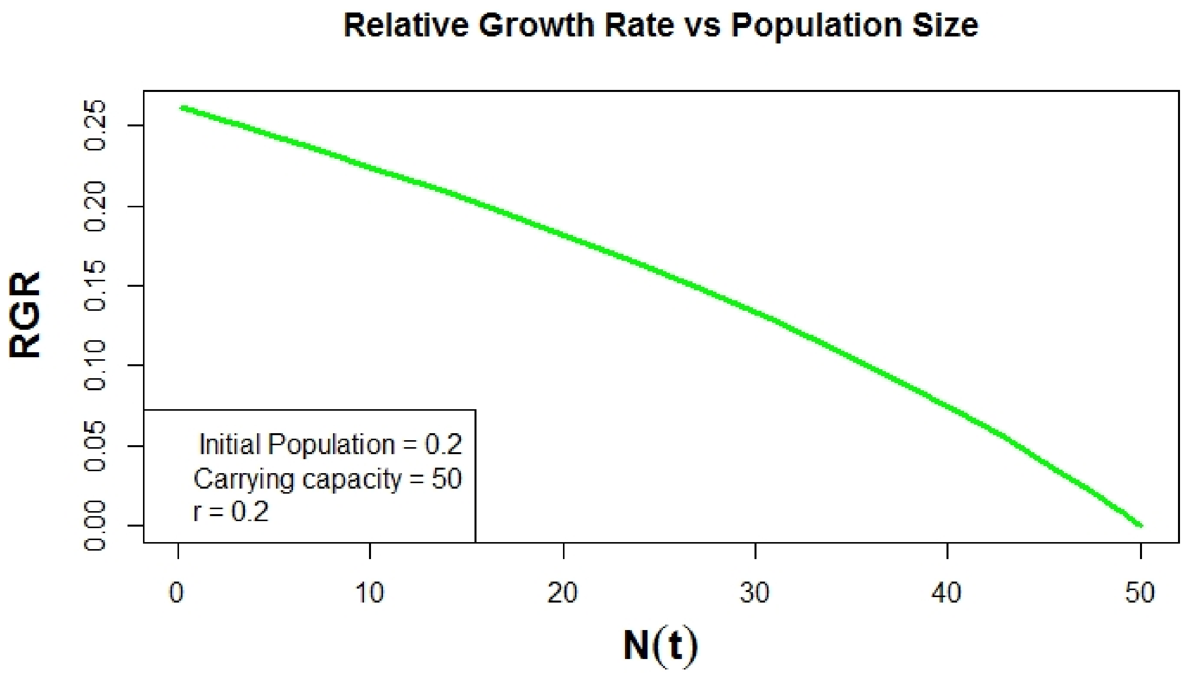

Figure 1: Negative density dependence for the logistic model

Ecologically RGR represents the proxy of species fitness. Therefore, with increasing population size, species fitness is decreased linearly, which in-turn is responsible for the negative density dependence of the species fitness. But in most of the cases this relationship is not linear yet.

2.2 Theta- Logistic Model

Sibly et. al[2] has shown that most of the species does not maintain this linear relationship between any population and its fitness. During his investigation, he choose a model proposed by Gillipin et. al.[3] during his fruit fly experiment, popularly known as θ-logistic model. This model is given by

$$\frac{1}{N(t)}\frac{dN(t)}{dt} = r\left(1 - \left(\frac{N(t)}{K}\right)^{\theta}\right) \tag{3}$$

This model differs from the logistic model (2) due to the introduction of the parameter θ. This parameter is responsible for maintaining the convex and concave relationship between species fitness and population, which is clearly visible from the Figure (2).

2.2.1 *Mathematical Solution*

The θ-logistic model is given by

$$\frac{dN}{dt} = rN(t)[1 - (\frac{N(t)}{k})^{\theta}] \tag{4}$$

This equation can be written as

$$dt = \frac{dN(t)}{rN(t)[1-(\frac{N^{\theta}(t)}{k^{\theta}})]}$$

$$or,\ rdt = \frac{k^{\theta} dN(t)}{N(t)[k^{\theta} - N^{\theta}(t)]}$$

$$or,\ rdt = \left[\frac{1}{N(t)} + \frac{N^{(\theta-1)(t)}}{k^{\theta} - N^{\theta}(t)}\right] dN(t)$$

Integrating with respect to 't' both sides we get

$$rt = \ln\frac{N(t)}{N(0)} - \frac{1}{\theta}\ln\frac{k^{\theta} - N^{\theta}(t)}{k^{\theta} - N^{\theta}(0)}$$

$$or,\ rt = \ln\frac{N(t)}{N(0)}[\frac{k^{\theta} - N^{\theta}(0)}{k^{\theta} - N^{\theta}(t)}]^{\frac{1}{\theta}}$$

$$or,\ \frac{N(t)}{[k^{\theta} - N^{\theta}(t)]^{\frac{1}{\theta}}} = \frac{N(0)}{[k^{\theta} - N^{\theta}(0)]^{\frac{1}{\theta}}} e^{rt} = ce^{rt}$$

where $c = \frac{N(0)}{[k^{\theta} - N^{\theta}(0)]^{\frac{1}{\theta}}}$

Now, if $0 < N(0) < k$ then $c^{\theta} = \frac{N^{\theta}(0)}{k^{\theta} - N^{\theta}(0)} > 0$ as $N^{\theta}(0) < k^{\theta}$

Therefore

$$\frac{N^{\theta}(t)}{k^{\theta} - N^{\theta}(t)} = \frac{N^{\theta}(0)}{k^{\theta} - N^{\theta}(0)} e^{\theta rt}$$

$$or,\ N^{\theta}(t) = \frac{N^{\theta}(0)(k^{\theta} - N^{\theta}(t))}{k^{\theta} - N^{\theta}(0)} e^{\theta rt}$$

$$or,\ N(t) = [\frac{k^{\theta}(N^{\theta}(0))}{N^{\theta}(0) + (K^{\theta} - N^{\theta}(0))e^{-\theta rt}}]^{\frac{1}{\theta}}$$

Which is the exact solution of the θ-logistic model.

2.2.2 Role Of The Curvature Parameter θ Towards The Ecosystem

Like, logistic model as discussed above, theta-logistic model (3) can also be written as following

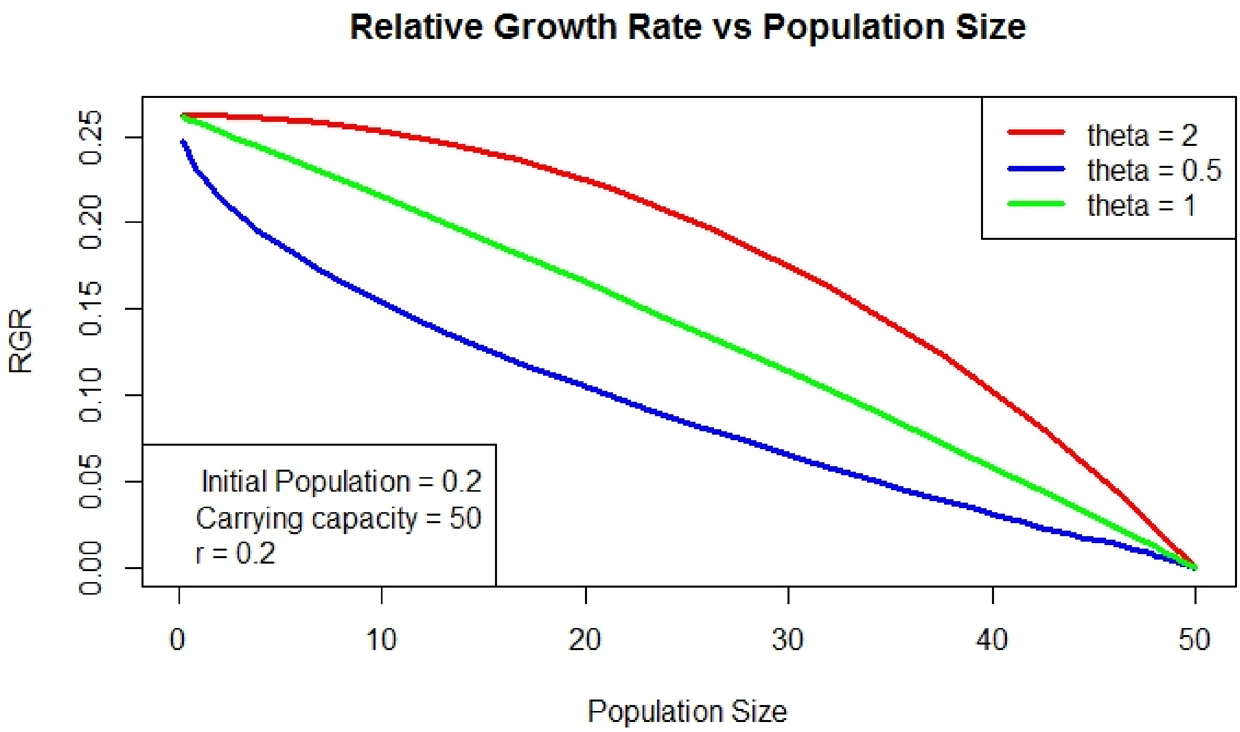

Figure 2: Negative density dependence for the theta - logistic model

$$R(t) = r\left(1 - \left(\frac{N(t)}{K}\right)^{\theta}\right)$$

$$R(t) = r - \frac{rN^{\theta}(t)}{K^{\theta}}$$

$$R(t) + \frac{rN^{\theta}(t)}{K^{\theta}} = r$$

$$\frac{R(t)}{r} + \frac{N^{\theta}(t)}{K^{\theta}} = 1$$

• Here $\theta = 1$ represents the logistic growth.

• Since, the quantity $\frac{N(t)}{K}$ is always less than 1, so for $\theta > 1$ there is a tendency to move away from the linear graph i.e. there is a convex downward relationship between population fitness and their size.

• Again, if $\theta < 1$, there must be a deviation from the linear graph and that's why the curve will move towards the origin. This is the reason behind the concave downward relationship between species fitness and population.

This phenomena is illustrated through the Figure 2.

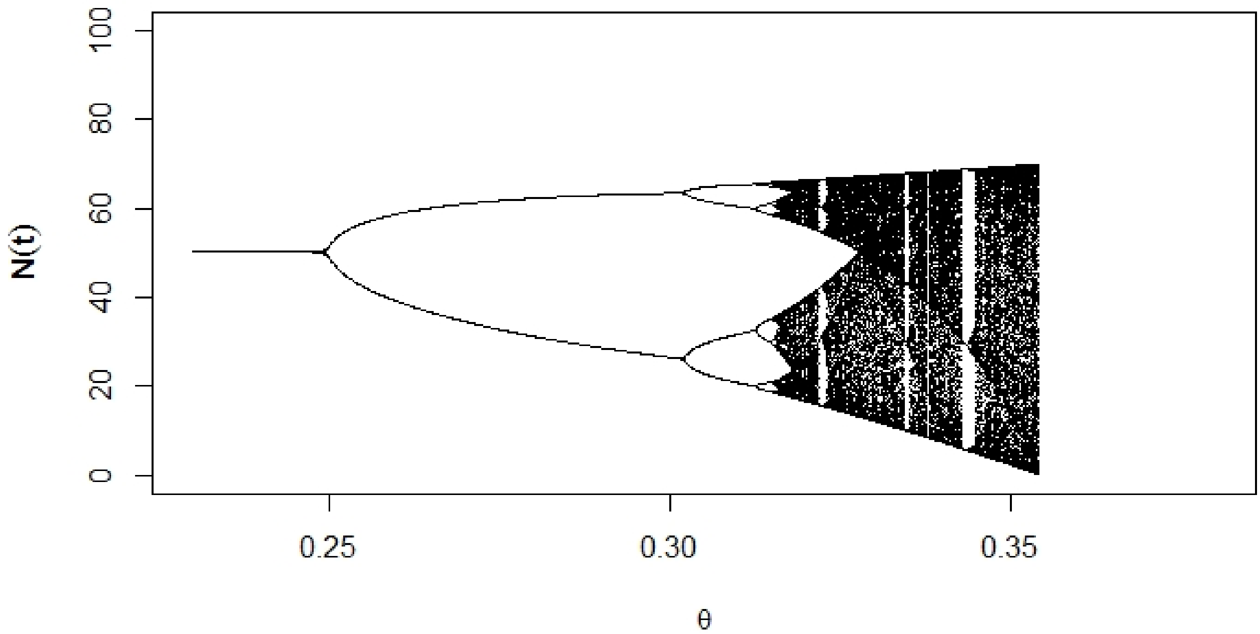

Figure 3: Bifurcation analysis of the θ logistic model

2.2.3 *Bifurcation analysis through the parameter* θ

In growth curve literature, equilibrium point plays a major role in analyzing species sustainability in the ecosystem. Equilibrium point means those point at which the absolute growth rate (henceforth, AGR) is zero. Here the model (3) has two equilibrium points, one is N(t) = 0 and another is N(t) = K. Here nature of the equilibrium point N(t) = 0 is unstable and that of N(t) = K asymptotically stable equilibrium point.

Due to the presence of the different parameters in the growth curve model, there must be a serious impact of the parameter towards their stability. This phenomenon can be demonstrated through the bifurcation.

Bifurcation is nothing but the qualitative change of the equilibrium point from stable to unstable one. This happens due to the variation of any parameter. Here, we consider the parameter θ as the bifurcation parameter. This bifurcation analysis is demonstrated through the Figure (3).

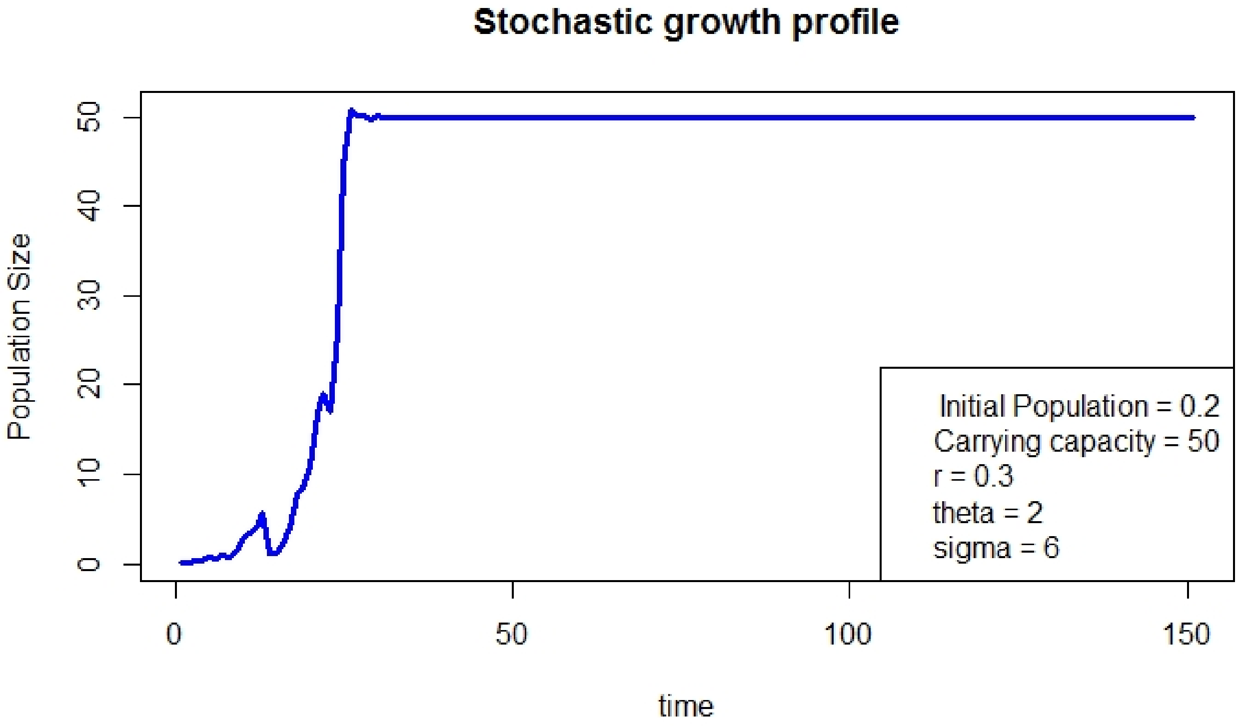

Figure 4: Stochastic growth profile for theta logistic model

3 STOCHASTIC ANALOGUE OF THE THETA-LOGISTIC MODEL

Population growth models are abstract representation of the real world objects, systems or processes to illustrate the theoretical concepts that these days are increasingly being used in more applied situations such as predicting future outcomes or simulation experiments. In mathematical literature[4, 5], many population models have been considered, from deterministic and stochastic population models where the population size is represented by a discrete random variable, to very complex continuous stochastic models. A nonrandom case, ignores natural variation and produces a single value result, whereas a stochastic model incorporates some natural variations in to model to state unpredictable situations such as weather or random fluctuations in resources and it generates a mean or most probable result. Nowadays, the well-known model like logistic is playing a major role in modern ecological theory. Due to incorporation of stochasticity species growth profile will be no longer smooth. This is demonstrated through the Figure 4.

4 RESULTS AND DISCUSSIONS:

The environmental perturbation hampers the reproductive rate of any species. Thus, the offspring wil not produce in a regular manner. Therefore, the system becomes a stochastic one. This randomness can be introduced into the system through additively or through any model parameters. Here we consider the additive noise into the system, then this scenario can be described through the stochastic differential equation[6]

$$dN(t) = rN(t)\left(1 - \left(\frac{N(t)}{K}\right)^{\theta}\right)dt + \sigma(N(t), t)dW(t)$$

Where, $\sigma(N(t), t)$ be the environmental noise intensity. Moreover, W(t) represents Wiener process with mean zero and variance of dW(t) is dt.

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