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Common Fixed Point Theorem of Weakly Compatible Maps In Complete Metric Space

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ABSTRACT

In this paper we prove common fixed point theorem of three maps using weakly compatible mappings in complete Metric space. We prove Common fixed point results of pair of weakly compatible maps.

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KEYWORDS:-Contraction mapping, Fixed point, Common fixed point, weakly compatible mapping.

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1 . INTRODUCTION

The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research activity during the some decades.

The fixed point theory has several applications in many fields of science and engineering .S. Banach[1]derived a well known theorem for a contraction mapping in a complete Metric space, which states that, “A contraction mapping has a unique fixed point in a complete Metric space. After that several authors proved fixed point theorems for mappings satisfying certain contraction conditions. In 1968 R. Kanan [2]introduced another type of map called as Kanan map and obtained unique fixed point theorem in complete Metric space. In 1973 B.K. Das and Sattya Gupta [3]Generalized Banach Contraction Principle in terms of rational expression in Complete Metric space .Various fixed point thereoms proved by many authors. Recently Latpate V.V. and Dolhare U.P.[4]proved fixed point theorems for uniformly locally contractive mappings.

Sesa[5]Introduced a concept of weakly commuting mappings and proved some common fixed point theorems in complete Metric Space.PatilS.T.[6]Proved some common fixed point theorems for weakly commuting mappings satisfying a contractive conditions in complete Metric space. In 1986 G. Jungck[7]defined notion of compatible mappings and proved some common fixed point theorems in complete Metric space. Also he proved weak commuting mappings are compatible.

Also we prove the common fixed point theorem of weakly compatible maps satisfying the inequality similar to C- Contraction.

2. PRELIMINARY DEFINITIONS AND EXAMPLES

Definition 2.1:-Let X be a non-emptiest . A mapping $d : X \times X \to R$ is said to be a Metric or a distance function if it satisfies following conditions.

1 . $d(x, y)$ is non-negative.

2 . $d(x, y)=0$ if and only if x and y coincides i.e. x=y.

3. $d(x, y) = d(y, x)$ (Symmetry)

4. $d(x, y) \le d(x, z) + d(z, y)$ (Triangle inequality)

Then the function d is said to be ametric on X. And (X, d) or simply X is said to as Metric space.

Definition 2.2:- A Metric space (X, d) is said to be a complete Metric space if every Cauchy sequence in X converges to a point of X .

Definition 2.3:- If (X, d) be a complete Metric space and a function $F : X \to X$ is said to be a contraction map if

$$d(F(x), F(y)) \le \beta d(x, y)$$

For all $x, y \in X$ and for $0 < \beta < 1$

Definition 2.4:- Let $F : X \to X$, then $x \in X$ is said to be a fixed point of F if $F(x) = x$

Definition 2.5:-Let X be a Metric space and if F_1 *and* F_2 be any two maps. An element $a \in X$ is said to be a common fixed point of F_1 *and* F_2 if $F_1(a) = F_2(a)$

For ex:- If $F_1(x) = sin(x)$ *and* $F_2(x) = tan(x)$

Then 0 is the common fixed point F_1 *and* F_2 *Since* $F_1(0) = sin(0)$ *and* $F_2(0) = tan(0) = 0$

Definition 2.6:-(S.K. Chatterjea) [8]

A mapping $F : X \to X$ where (X, d) is a Metric space is said to be C-Contraction if there is a some β s.t. $0 < \beta < \frac{1}{2}$ s.t. the following inequality holds

$$d(F_x, F_y) \leq \beta(d(x, F_y) + d(y, F_x))$$

If (X, d) be a complete Metricspace, then any C-contraction on X has a unique fixed point.

Definition 2.7:- Let F and G be two self mappings of a Metricspace (X, d), F and G are said to be weakly compatible if for all $x \in X$

$$F_x = G_x \Rightarrow FG_x = GF_x$$

Theorem 2.1:-Suppose P,Q,R,S be four self maps of a Metric space (X, d) which satisfies the conditions given below.

1. $P(X) \subseteq S(X)$ and $R(X) \subseteq Q(X)$.
2. Pair of mappings (P, Q) and (R, S) are Commuting.
3. One of the function P, Q, R, S is continuous.
4. $d(Px, Rx) \leq \mu\alpha(x, y)$ where $\alpha(x, y) = max\ d(Qx, Sy), d(Qx, Px), d(Sy, Ry)$

   For all $x, y \in X$ *and* $0 \leq \mu < 1$ and
5. X is complete.

Then P,Q,R and S have Unique Common Fixed point $z \in X$ .Furthermore z is the unique common fixed point of (P, Q) and (R, S).

Theorem 2.2:- Let (X, d) be a Complete Metric space. Suppose that the mappings P,Q,R and S are four self maps of X which satisfies the following,

1 S(X) $\subseteq P(x)$ *and* $R(X) \subseteq Q(X)$;

2 d(Rx,Sx) $\leq \psi(\alpha(x, y))$

Where ψ is an upper semi continuous, contractive modulus and

$$\alpha(x,y) = max\ \{d(Px,Qy), d(Px,Rx), d(Qy,Sy), \frac{1}{2}(d(Px,Sy)+d(Qy,Rx))\}$$

3. The pairs (R,P) and (S,Q) are weakly compatible. Then P,Q,R and S have a unique common fixed point.

We obtain Common Fixed point theorems for three maps which satisfiesfollowing contraction condition.

3 . MAIN RESULT

Theorem 3.1:- Let (X,d) be a complete Metric space and Let A be a non empty closed subset of X . Let $P,Q: A \to A$ be s.t.

$$d(P_x,Q_y) \le \frac{1}{2}(d(R_x,Q_y)+d(R_y,P_x)+d(S_x,R_y)) - \psi(d(R_x,Q_y)+d(R_y,P_x)) \quad (1.1)$$

For any $(x,y) \in X \times X$, where a function $\psi:[0,\infty)^2 \to [0,\infty)$ is a continuous and $\psi(x,y)=0$ iff $x=y=0$ and $R: A \to X$ which satisfies the following condition.

(i) PA $\subseteq$ RA and QA $\subseteq$ RA

(ii) The pair of mappings (P,R) and (Q,R) are weakly compatible.

(iii) R(A) is closed subset of X .

Then P ,R and Q have unique common fixed point.

Proof:- Let x_0 be any element of A as PA $\subseteq$ RA and QA $\subseteq$ RA.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences s.t. $y_0 = Px_0 = Rx_1, y_1 = Qx_1 = Rx_2, y_2 = Px_2 = Rx_3, \ldots\ldots\ldots$
$\ldots\ldots\ldots y_{2n} = Px_{2n} = Rx_{2n+1},\ y_{2n+1} = Qx_{2n+1} = Rx_{2n+2}, = \ldots\ldots\ldots$

First we shall prove that $d(y_n, y_{n+1}) \to 0$ as $n \to \infty$

Let $n = 2k$ by inequality (1), we have

$$d(y_{2n+1}, y_{2k}) = d(Px_{2k}, Qx_{2k+1})$$

$$\le \frac{1}{2}(d(Rx_{2k},Qx_{2k+1}) + d(Rx_{2k+1},Px_{2k}) + d(Sx_{2k},Rx_{2k+1})) - \psi(d(Rx_{2k},Qx_{2k+1}), d(Rx_{2k+1},Px_{2k}))$$

$$= \frac{1}{2}(d(y_{2k-1},y_{2k+1}) + d(y_{2k},y_{2k}) + d(y_{2k},y_{2k})) - \psi(d(y_{2k-1},y_{2k+1}), d(y_{2k},y_{2k}))$$

$$\le \frac{1}{2}(d(y_{2k-1},y_{2k+1}) \quad (1.2)$$

$$\leq \frac{1}{2}(d(y_{2k-1}, y_{2k}) + d(y_{2k}, y_{2k+1}))$$

This gives

$$d(y_{2k+1}, y_{2k}) \leq d(y_{2k}, y_{2k-1})$$

For $n = 2k+1$, similarly, we can show that

$$d(y_{2k+2}, y_{2k+1}) \leq d(y_{2k+1}, y_{2k}) \tag{1.3}$$

$d(y_{n+1}, y_n)$ is a non-increasing sequence of non-negative real numbers and hence it is convergent .

Let $l = \lim_{n\to\infty} d(y_{n+1}, y_n)$ From (1.2) we have

$d(y_{n+1}, y_n) \leq \frac{1}{2} d(y_{n-1}, y_{n+1})$ and by triangle inequality

$$d(y_{n+1}, y_n) \leq \frac{1}{2}(d(y_{n-1}, y_n) + d(y_n, y_{n+1})) \tag{1.4}$$

Let $n \to \infty$, we have

$$\lim_{n\to\infty} d(y_{n+1}, y_n) \leq \frac{1}{2} \lim_{n\to\infty} d(y_{n-1}, y_{n+1}) \leq \lim_{n\to\infty} d(y_{n+1}, y_n)$$

$$l \leq \frac{1}{2} \lim_{n\to\infty} d(y_{n-1}, y_{n+1}) \leq l$$

$$\lim_{n\to\infty} d(y_{n-1}, y_{n+1}) = 2l$$

Consider

$$d(y_{2k+1}, y_{2k}) = d(Px_{2k}, Qx_{2k+1})$$

$$\leq \frac{1}{2}(d(y_{2k-1}, y_{2k+1}) + d(y_{2k}, y_{2k}) + d(y_{2k}, y_{2k})) - \psi(d(y_{2k-1}, y_{2k+1}), d(y_{2k}, y_{2k})) \tag{1.5}$$

Let $k \to \infty$ and Since ψ is given to be continuous $\therefore$ we get

$$l \leq \frac{1}{2} 2l - \psi(2l, 0)$$

This gives $\psi(2l, 0) = 0$

By definition of ψ, $\psi(x, y) = 0$ *if* $x = y = 0$

$\therefore 2l = 0$, $\therefore l = 0$

$$l = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0 \tag{1.5}$$

Now our claim is that $\{y_n\}$ is a Cauchy sequence

From (1.3) we have

$d(y_{n+1}, y_{n+2}) \le d(y_n, y_{n+1})$,

To prove $\{y_n\}$ is a Cauchy sequence we only prove that the subsequence $\{y_{2n}\}$ is a Cauchy sequence.

If possible suppose that $\{y_{2n}\}$ is not a Cauchy sequence.

There exists $\delta > 0$ for which we can find two subsequence's $\{y_{2n(k)}\}$ and $\{y_{2m(k)}\}$ of $\{y_{2n}\}$

Such that n_k is the least index for which $n_k > m_k >$k and d($y_{2m(k)}, y_{2n(k)}) \ge \delta$

This gives

$$d(y_{2m(k)}, y_{2n(k)-2}) < \delta \tag{1.6}$$

Using triangle inequality

$$\delta \le d(y_{2m(k)}, y_{2n(k)}) \le d(y_{2m(k)}, y_{2n(k)-2}) + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}) \tag{1.7}$$

Now as $k \to \infty$ and from (1.6), we have

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) = \delta \tag{1.8}$$

$$|d(y_{2m(k)}, y_{2n(k)+1}) - d(y_{2m(k)}, y_{2n(k)})| \le d(y_{2n(k)}, y_{2n(k)+1}) \tag{1.9}$$

Also

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \le d(y_{2m(k)}, y_{2m(k)-1}) \tag{1.10}$$

And

$$|d(y_{2n(k)}, y_{2m(k)-2}) - d(y_{2n(k)}, y_{2m(k)-1})| \le d(y_{2m(k)-2}, y_{2m(k)-1}) \tag{1.11}$$

From (1.6),(1.9),(1.10) and (1.12) , We have

$$\lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)}) = \lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)-1}) = \lim_{k\to\infty} d(y_{2m(k)-2}, y_{2n(k)}) = \grave{o} \tag{1.12}$$

Inequality (1.1) gives

$$d(y_{2m(k)-1}, y_{2n(k)}) = d(Px_{2n(k)}, Qx_{2m(k)-1})$$

$$\leq \frac{1}{2}(d(Rx_{2n(k)}, Qx_{2m(k)-1}) + d(Rx_{2m(k)-1}, Px_{2n(k)}) + d(Sx_{2n(k)}, Rx_{2m(k)-1}))$$
$$- \psi(d(Rx_{2n(k)}, Qx_{2m(k)-1}), d(Rx_{2m(k)-1}, Px_{2n(k)})$$

$$= \frac{1}{2}d(y_{2n(k)-1}, y_{2m(k)-1}) + d(y_{2m(k)-2}, y_{2n(k)}) + d(y_{2n(k)}, y_{2m(k)-2})) -$$
$$\psi(d(y_{2n(k)-1}, y_{2m(k)-1}), d(y_{2m(k)-2}, y_{2n(k)}))$$

$$\leq \frac{1}{2}(d(y_{2m(k)-1}, y_{2m(k)}) + d(y_{2m(k)}, y_{2m(k)+1})) \qquad (1.13)$$

Let $k \to \infty$ in above inequality and from (1.13) and ϕ is continuous, $\therefore$ we get

$$\delta \leq \frac{1}{2}(\delta + \delta) - \psi(\delta + \delta)$$

$\therefore$ This gives $\psi(\delta, \delta) = 0$. By assumption of $\phi(x, y) = 0$ *if* $x = y = 0$

$\therefore \delta = 0.$ *But* $\delta > 0$

therefore which givescontradiction. $\therefore$ $\{y_n\}$ is a Cauchy Sequence.

To prove P,Q,R have a Common fixed point. Given (X, d) be complete and $\{y_n\}$ be Cauchy Sequence ,

$\therefore$ there is $p \in X$ s.t. $\lim_{n\to\infty} y_n = p$ as A is closed and $\{y_n\} \subseteq A$, $\therefore$ $p \in A$, By hypothesis $R(A)$ is closed .So there is $u \in A$ s.t. $p = Ru$ for every $n \in N$

$$d(Pu, y_{2n+1}) = d(Pu, Qx_{2n+1})$$
$$\leq \frac{1}{2}(d(Ru, Qx_{2n+1}) + d(Rx_{2n+1}, Pu) + d(Su, Rx_{2n+1})) \qquad (1.14)$$
$$- \psi(d(Ru, Qx_{2n+1}), d(Rx_{2n+1}, Pu))$$

$$= \frac{1}{2}(d(p, y_{2n+1}) + d(y_{2n}, Pu) + d(Su, y_{2n}))$$
$$-\psi(d(Ru, Qx_{2n+1}), d(Rx_{2n+1}, Pu))$$

When $n \to \infty$

$$d(Pu, p) \leq \frac{1}{2}(d(p, p) + d(p, Pu) + d(Su, p)) - \psi(d(Ru, p), d(p, Pu))$$

And hence

$$\psi(0, d(p, Pu)) \le -\frac{1}{2}(d(Pu, p) + d(Su, p)) \le 0,$$

$$\therefore d(p, pu) = 0 \therefore Pu = p$$

Simillarly we can show that $Su = p. \ \therefore Pu = Qu = Ru = p.$

Given pairs (R, P) and (R, Q) are weakly compatible ,$\therefore Pp = Qp = Rp$

Now consider

$$d(Pp, y_{2n+1}) = d(Pp, Qx_{2n+1})$$

$$\le \frac{1}{2}(d(Rp, Qx_{2n+1}) + d(Rx_{2n+1}, Pp) + d(Sp, Rx_{2n+1})) \qquad (1.15)$$

$$- \psi(d(Rp, Qx_{2n+1}), d(Rx_{2n+1}, Pp))$$

$$= \frac{1}{2}(d(Rp, y_{2n+1}) + d(y_{2n}, Pp) + d(Sp, y_{2n}))$$

$$-\psi(d(Rp, y_{2n+1}), d(y_{2n}, Pp))$$

As $n \to \infty$, Since $Pp = Qp = Rp$, We have

$$d(Pp, p) \le \frac{1}{2}(d(Pp, p) + d(p, Pp) + d(Sp, p)) - \psi(d(Pp, p), d(p, Pp)) \quad (1.16)$$

Hence $\psi(d(Pp, p), d(p, Pp)) = 0$ and so $d(Pp, p) = 0$

$\therefore \ Pp = p$ *and from* $Pp = Qp = Rp$

We get $Pp = Qp = Rp = p$.

Thus Uniqueness of common fixed point is easily obtained from inequality (1.1)

CONCLUSION:- Thus we proved common fixed point theorem for weakly compatible mappings.

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