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Generalized ℓ-Difference Operator with Two Variables

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ABSTRACT

The difference equations manifest themselves as mathematical models describing real life situations in probability theory, queuing problems, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical networks, genetics in biology, economics, psychology, sociology, etc., In this paper, we discuss results on ℓ -difference operator with two variable and its inverse

KEYWORDS: ℓ-difference equation, ℓ-difference operator, difference equation

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1.INTRODUCTION

The theory of difference equations has grown tremendously during the last several years and it now occupies the central position in applicable analysis and will no doubt continue to play a vital and inevitable role in mathematics as a whole in the future as well, The basic theory of di□erence equations is based on the operator Δ defined as

Δu(k) =u(k+1)-u(k), k∈ N={0,1,2,3.....}.Eventhough many authors have suggested the definition of Δ as Δu(k)=u(k+ l)-u(k), k∈ R, l∈ $R-\{0\}$, (1.1)

But recently, E. Thandapani, M. M. S. Manuel, G.B. A. Xavier [6] considered the definition of Δ as given in (1.1) and developed the theory of di□erence equations in a di□erent direction. For convenience, the operator Δ defined by (1.1) is labelled as $\Delta_{(l)}$ and by defining its inverse $\Delta_{(l)}^{-1}$ many interestin results and application in number theory were obtained.

Definition: 1.1 Let $v(k)$ be a real valued function on (-∞,∞) and $l \neq 0$. Then the l-difference operator, denoted by $\Delta_{(l)}$, on $v(k)$ is defined as

$$\Delta_{(l)}\, v(k) = v(k+l) - v(k) \tag{1.2}$$

and the inverse of the l- difference operator, denoted by $\Delta_{(l)}^{-1}$, on $v(k)$ is defined as

$$\text{if } \Delta_{(l)}\, v(k) = u(k), \text{ then } v(k) = \Delta_{(l)}^{-1}\, u(k) \tag{1.3}$$

Proportion: 1.2 Let (e^k) be a real valued function on (-∞,∞) and $l \neq 0$. Then we have

$$(\boldsymbol{i})\ {\Delta_l}^n(e^k) = e^k(e^l-1)^n \tag{1.4}$$

$$(\boldsymbol{ii})\ {\Delta_l}^n(e^{k+l}) = e^{k+l}(e^l-1)^n \tag{1.5}$$

$$(\boldsymbol{iii})\ {\Delta_l}^n(e^{k+nl}) = e^{k+nl}(e^l-1)^n \tag{1.6}$$

$$(\boldsymbol{iv})\ {\Delta_l}^{-n}(e^k) = \frac{e^k}{(e^l-1)^n} \tag{1.7}$$

$$(\boldsymbol{v})\ {\Delta_l}^{-n}(e^{k+l}) = \frac{e^{k+l}}{(e^l-1)^n} \tag{1.8}$$

$$(\boldsymbol{vi})\ {\Delta_l}^{-n}(e^{k+nl}) = \frac{e^{k+nl}}{(e^l-1)^n} \tag{1.9}$$

Proof:

(i) By definition

$$\Delta_l(e^k) = e^{k+l} - e^k = e^k(e^l-1)$$

$${\Delta_l}^2(e^k) = e^k(e^l-1)(e^l-1) = e^k(e^l-1)^2$$

$${\Delta_l}^3(e^k) = e^k(e^l-1)^2(e^l-1) = e^k(e^l-1)^3$$

Continuing this process we get the result

(ii),(iii) The proof is similar to (i)

(iv) By definition

$$\Delta_l(e^k) = e^k(e^l-1)$$

$$\Delta_l^{-1}(e^k) = \frac{e^k}{(e^l-1)}$$

$$\Delta_l^{-2}(e^k) = \frac{e^k}{(e^l-1)(e^l-1)} = \frac{e^k}{(e^l-1)^2}$$

$$\Delta_l^{-3}(e^k) = \frac{e^k}{(e^l-1)^2\,(e^l-1)} = \frac{e^k}{(e^l-1)^3}$$

Continuing this process we get the result.

(v),(vi) The proof is similar to (iv)

Theorem :1.3 For the real valued function u(k) and v(k) then

$$\Delta_l^{n}(klog\ k) = \sum_{j=0}^{n}(-1)^j\,[nc_j(k+(n-j)l)\log(k+(n-j)l)] \tag{1.10}$$

Proof: By definition

$$\Delta_l(klogk) = (k+l)\log(k+l) - klogk$$

$$\Delta_l^{2}(klog\ k) = \Delta_l\big(\Delta_l(klogk)\big)$$

$$= (k+2l)\log(k+2l) - (k+l)\log(k+l) - (k+l)\log(k+l) + klogk$$

$$= (k+2l)\log(k+2l) - 2(k+l)\log(k+l) + klogk$$

$$\Delta_l^{3}(klog\ k) = \Delta_l(\Delta_l^{2}(klog\ k))$$

$$= (k+3l)\log(k+3l) - 3(k+2l)\log(k+2l) + 3\,(k+l)\log(k+l) - klogk$$

$$\Delta_l^{4}(klog\ k) = \Delta_l(\Delta_l^{3}(klog\ k))$$

$$= (k+4l)\log(k+4l) - 4(k+3l)\log(k+3l) + 6(k+2l)\log(k+2l)$$

$$-4(k+l)\log(k+l) + klogk$$

Proceeding like this, we get proof of this theorem.

Corolary:1.4 For the real valued function u(k) and v(k) then

$$\Delta_l^{n}(k(logk)^5) = \sum_{j=0}^{n}(-1)^j\,[nc_j(k+(n-j)l)log(k+(n-j)l)^5]$$

Proof: The proof follows by taking $logk = (logk)^5$in theorem (1.3)

Corolary:1.5 For the real valued function u(k) and v(k) then

$$\Delta_l^{n}(k(logk)^t) = \sum_{j=0}^{n}(-1)^j\,[nc_j(k+(n-j)l)\ log(k+(n-j)l)^t]$$

Proof: The proof follows by taking $logk = (logk)^t$ in (1.3)

Theorem :1.6 For the real valued function u(k) and v(k) we have

$$\Delta_l^{n}(ke^k) = (e^l-1)^{n-1}[ke^k(e^l-1) + nle^{k+l}] \tag{1.11}$$

Proof: By definition

$$\Delta_l(ke^k) = (k+l)e^{k+l} - ke^k$$

$$= ke^k(e^l - 1) + le^{k+l}$$

$$\Delta_l{}^2(ke^k) = \Delta_l\left(\Delta_l(ke^k)\right)$$

$$= (e^l - 1)(\ ke^k(e^l - 1) + le^{k+l}) + le^{k+2l} - le^{k+l}$$

$$= (e^l - 1)[\ ke^k(e^l - 1) + 2le^{k+l}]$$

$$\Delta_l{}^3(ke^k) = \Delta_l(\Delta_l{}^2(ke^k))$$

$$= (e^l - 1)[(e^l - 1)\ ke^k(e^l - 1) + le^{k+l} + 2\ le^{k+2l} - 2le^{k+l}]$$

$$= (e^l - 1)^2[ke^k(e^l - 1) + 3le^{k+l}]$$

$$\Delta_l{}^4(ke^k) = \Delta_l(\Delta_l{}^3(ke^k))$$

$$= [(e^l - 1)^2[(e^l - 1)\ ke^k(e^l - 1) + le^{k+l} + 3\ le^{k+2l} - 3le^{k+l}]$$

$$= (e^l - 1)^3[ke^k(e^l - 1) + 4le^{k+l}]$$

Continuing like this, we get proof of this theorem.

Theorem :1.7 For the real valued function u(k) and v(k) we have

$$\Delta_l{}^{-n}(ke^k) = \frac{1}{(e^l-1)^{n+1}}[ke^k(e^l - 1) - nle^{k+l}] \tag{1.12}$$

Proof: By product formula,

$$\Delta_l{}^{-1}(u(k)v(k)) = u(k)\Delta_l{}^{-1}(v(k)) - \Delta_l{}^{-1}[\Delta_l{}^{-1}(v(k + l))\Delta_l u(k)]$$

$$\Delta_l{}^{-1}(ke^k) = k\Delta_l{}^{-1}(e^k) - \Delta_l{}^{-1}[\Delta_l{}^{-1}(e^{k+l})\Delta_l(k)]$$

$$\Delta_l{}^{-1}(ke^k) = \frac{ke^k}{(e^l - 1)} - \frac{1}{(e^l - 1)}\ \Delta_l{}^{-1}(le^{k+l})$$

$$= \frac{ke^k}{(e^l - 1)} - \frac{le^{k+l}}{(e^l - 1)^2}$$

$$= \frac{1}{(e^l - 1)^2}[ke^k(e^l - 1) - le^{k+l}]$$

$$\Delta_l{}^{-2}(ke^k) = \frac{1}{(e^l - 1)^2}\left\{(e^l - 1)\frac{1}{(e^l - 1)^2}[ke^k(e^l - 1) - le^{k+l}] - \frac{le^{k+l}}{(e^l - 1)}\right\}$$

$$= \frac{1}{(e^l - 1)^3}[ke^k(e^l - 1) - 2le^{k+l}]$$

$$\Delta_l{}^{-3}(ke^k) = \frac{1}{(e^l - 1)^3}\left\{(e^l - 1)\frac{1}{(e^l - 1)^2}[ke^k(e^l - 1) - le^{k+l}] - \frac{2le^{k+l}}{(e^l - 1)}\right\}$$

$$= \frac{1}{(e^l - 1)^4}[ke^k(e^l - 1) - 3le^{k+l}]$$

Repeatedly like this, we get proof of this theorem

Theorem :1.8 If n is a Positive integer and $l_i>0$, then

$$\Delta_{l_n}\left(\Delta_{l_{n-1}}\Delta_{l_{n-2}} \dots\dots\dots\dots\dots\dots..\Delta_{l_2}\Delta_{l_1}e^k\right) = e^k\left(\prod_{i=1}^{n}(e^{l_i} - 1)\right) \tag{1.13}$$

Proof: By definition

$$\Delta_{l_1}(e^k) = e^{k+l_1} - e^k = e^k(e^{l_1} - 1)$$

$$\Delta_{l_2}(\Delta_{l_1}e^k) = e^k(e^{l_1} - 1)(e^{l_2} - 1)$$

$$\Delta_{l_3}(\Delta_{l_2}\Delta_{l_1}e^k) = e^k(e^{l_1} - 1)(e^{l_2} - 1)(e^{l_3} - 1)$$

$$\Delta_{l_4}(\Delta_{l_3}\Delta_{l_2}\Delta_{l_1}e^k) = e^k(e^{l_1} - 1)(e^{l_2} - 1)(e^{l_3} - 1)(e^{l_4} - 1)$$

Proceeding like this, we get proof of this theorem

Theorem :1.9 If n is a positive integer and $l_i>0$, then

$$\Delta_{l_n}{}^{-1}\left(\Delta_{l_{n-1}}{}^{-1}\Delta_{l_{n-2}}{}^{-1}\Delta_{l_{n-3}}{}^{-1} \dots\dots\dots.\Delta_{l_2}{}^{-1}\Delta_{l_1}{}^{-1}e^k\right) = \frac{e^k}{\prod_{i=1}^{n}(e^{l_i}-1)} \qquad (1.14)$$

Proof: By definition

$$\Delta_{l_1}{}^{-1}(e^k) = \frac{e^k}{(e^{l_1}-1)}$$

$$\Delta_{l_2}{}^{-1}(\Delta_{l_1}{}^{-1}e^k) = \frac{e^k}{(e^{l_1}-1)(e^{l_2}-1)}$$

$$\Delta_{l_3}{}^{-1}(\Delta_{l_2}{}^{-1}\Delta_{l_1}{}^{-1}e^k) = \frac{e^k}{(e^{l_1}-1)(e^{l_2}-1)(e^{l_3}-1)}$$

Continuing this way, we get proof of this theorem.

Theorem :1.10 If n is a Positive integer and $l_i>0$, then

$$\Delta_{l_n}\left(\Delta_{l_{n-1}}\Delta_{l_{n-2}} \dots\dots\dots\dots\dots\dots..\Delta_{l_2}\Delta_{l_1}ke^k\right)$$

$$= (\prod_{i=1}^{n} e^{l_i} - 1)ke^k + l_1(e^{l_2} - 1)(e^{l_3} - 1)\dots.(e^{l_n} - 1)e^{k+l_1}$$

$$+l_1(e^{l_2} - 1)(e^{l_3} - 1)\dots.(e^{l_n} - 1)e^{k+l_1} + l_2(e^{l_1} - 1)(e^{l_3} - 1)\dots.(e^{l_n} - 1)e^{k+l_2}$$

$$+ \cdots \dots\dots\dots\dots\dots..\dots\dots\dots\dots\dots\dots\dots\dots\dots. +l_n(e^{l_1} - 1)(e^{l_3} - 1)\dots.(e^{l_{n-1}} - 1)e^{k+l_n} \qquad (1.15)$$

Proof: By definition

$$\Delta_{l_1}(ke^k) = (e^{l_1} - 1)ke^k + l_1e^{k+l_1}$$

$$\Delta_{l_2}\left(\Delta_{l_1}ke^k\right) = (e^{l_1} - 1)(e^{l_2} - 1)ke^k + (e^{l_1} - 1)l_2e^{k+l_2} + (e^{l_2} - 1)l_1e^{k+l_1}$$

$$\Delta_{l_3}\left(\Delta_{l_2}\Delta_{l_1}ke^k\right) = (e^{l_1} - 1)(e^{l_2} - 1)(e^{l_3} - 1)ke^k + (e^{l_1} - 1)(e^{l_2} - 1)l_3e^{k+l_3}$$

$$+(e^{l_1} - 1)(e^{l_3} - 1)l_2e^{k+l_2} + (e^{l_2} - 1)(e^{l_3} - 1)l_1e^{k+l_1}$$

Proceeding like this, we get proof of this theorem

Theorem :1.11 If n is a positive integer and $l_i > 0$, then

$$\Delta_{l_n}{}^{-1}\left(\Delta_{l_{n-1}}{}^{-1}\Delta_{l_{n-2}}{}^{-1}\Delta_{l_{n-3}}{}^{-1}\dots\dots\dots\Delta_{l_2}{}^{-1}\Delta_{l_1}{}^{-1}ke^k\right) = \frac{ke^k}{\prod_{i=1}^{n}(e^{l_i}-1)}$$
$$-\frac{l_1e^{k+l_1}}{(e^{l_1}-1)^2(e^{l_2}-1)(e^{l_3}-1)\dots(e^{l_n}-1)}-\frac{l_2e^{k+l_2}}{(e^{l_1}-1)(e^{l_2}-1)^2(e^{l_3}-1)\dots(e^{l_n}-1)}$$
$$-\dots\dots\dots\dots\dots\dots-\frac{l_ne^{k+l_n}}{(e^{l_1}-1)(e^{l_2}-1)(e^{l_3}-1)\dots(e^{l_{n-1}}-1)(e^{l_n}-1)^2} \quad (1.16)$$

Proof: By definition

$$\Delta_{l_1}{}^{-1}(ke^k) = \frac{1}{(e^{l_1}-1)^2}\left[ke^k(e^{l_1}-1)-l_1e^{k+l_1}\right]$$

$$\Delta_{l_2}{}^{-1}(\Delta_{l_1}{}^{-1}ke^k) = \frac{1}{(e^{l_1}-1)^2}\left\{(e^{l_1}-1)\frac{1}{(e^{l_2}-1)^2}\left[ke^k(e^{l_2}-1)-l_2e^{k+l_2}\right]-\frac{l_1e^{k+l_1}}{(e^{l_2}-1)}\right\}$$

$$= \frac{ke^k}{(e^{l_1}-1)(e^{l_2}-1)}-\frac{l_1e^{k+l_1}}{(e^{l_1}-1)^2(e^{l_2}-1)}-\frac{l_2e^{k+l_2}}{(e^{l_1}-1)(e^{l_2}-1)^2}$$

$$\Delta_{l_3}{}^{-1}(\Delta_{l_2}{}^{-1}\Delta_{l_1}{}^{-1}ke^k) = \frac{1}{(e^{l_1}-1)(e^{l_2}-1)}\left\{\frac{1}{(e^{l_3}-1)^2}\left[ke^k(e^{l_3}-1)-l_3e^{k+l_3}\right]\right\}$$
$$-\frac{l_1e^{k+l_1}}{(e^{l_1}-1)^2(e^{l_2}-1)(e^{l_3}-1)}-\frac{l_2e^{k+l_2}}{(e^{l_1}-1)(e^{l_2}-1)^2(e^{l_3}-1)}$$

$$\Delta_{l_3}{}^{-1}(\Delta_{l_2}{}^{-1}\Delta_{l_1}{}^{-1}ke^k) = \frac{ke^k}{(e^{l_1}-1)(e^{l_2}-1)(e^{l_3}-1)}-\frac{l_1e^{k+l_1}}{(e^{l_1}-1)^2(e^{l_2}-1)(e^{l_3}-1)}$$
$$-\frac{l_2e^{k+l_2}}{(e^{l_1}-1)(e^{l_2}-1)^2\,(e^{l_3}-1)}-\frac{l_3e^{k+l_3}}{(e^{l_1}-1)(e^{l_2}-1)(e^{l_3}-1)^2}$$

Continuing this process, we get proof of this theorem

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