Investigation of Inter Relation Among Different Types of Continuous Functions on Convex Topological Space

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ABSTRACT

In this paper some continuous functions has been introduced using both the topology $\tau$ and convexity $\mathcal{C}$ on the same underlying set $X$ where $(X, \tau, \mathcal{C})$ is termed as convex topological space and inter relation among them are also investigated.

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1. INTRODUCTION

The development of “abstract convexity” has emanated from different sources in different ways; the first type of development basically banked on generalization of particular problems such as separation of convex sets\(^1\), extremality\(^2,3\) or continuous selection\(^4\). The second type of development lay before the reader such axiomatizations, which in every case of design, express particular point of view of convexity. With the view point of generalized topology which enters into convexity via the closure or hull operator, Schmidt and Hammer, introduced some axioms to explain abstract convexity. The arising of convexity from algebraic operations and the related property of domain finiteness receive attentions in Birchoff and Frink, Schmidt, Hammer.

The axiomatizations as proposed by M.L.J. Van De Vel in his paper Theory of Convex Structure\(^5\) will be followed through out in this paper.

The author has discussed in “Topology and Convexity on the same set”\(^6\) and introduced the compatibility of the topology with a convexity on the same underlying set. At the very early stage of this paper we have set aside this concept of compatibility and started just with a triplet \((X, \tau, C)\) and call it convex topological space only to bring back “compatibility” in another way subsequently. With this compatibility, Van De Vel has called the triplet \((X, \tau, C)\) a topological convex structure.

In this paper, Art. 2 deals with some early definitions, results and in Art. 3 we have discussed mainly inter relation among different types of continuous functions.

2. PREREQUISITES:

**Definition 2.1**\(^6\): Let \(X\) be a non empty set. A family \(C\) of subsets of the set \(X\) is called a convexity on \(X\) if

1. \(\phi, X \in C\)
2. \(C\) is stable for intersection, i.e. if \(D \subseteq C\) is non empty, then \(\cap D \in C\)
3. \(C\) is stable for nested unions, i.e. if \(D \subseteq C\) is non empty and totally ordered by set inclusion, then \(\cup D \in C\).

The pair \((X, C)\) is called a convex structure. The members of \(C\) are called convex sets and their complements are called concave sets.

**Definition 2.2**\(^6\): Let \(C\) be a convexity on set \(X\). Let \(A \subseteq X\). The convex hull of \(A\) is denoted by \(co(A)\) and defined by \(co(A) = \cap \{C : A \subseteq C \in C\}\).

**Note 2.3**\(^6\): Let \((X, C)\) be a convex structure and let \(Y\) be a subset of \(X\). The family of sets \(C_Y = \{C \cap Y : C \in C\}\) is a convexity on \(Y\); called the relative convexity of \(Y\).

**Note 2.4**\(^6\): The hull operator \(co_Y\) of a subspace \((Y, C_Y)\) satisfy the following:
∀A ⊆ Y : \text{co}_Y(A) = \text{co}(A) \cap Y.

**Definition 2.5** ⁶: Let \((X, \mathcal{C})\) be a convex structure and let \(\tau\) be a topology on \(X\). Then \(\tau\) is said to be compatible with the convex structure \((X, \mathcal{C})\) if all polytopes of \(\mathcal{C}\) are closed in \(\tau\) where polytopes means convex hull of a finite set. Also the triplet \((X, \tau, \mathcal{C})\) is then called topological convex structure.

**Note 2.6** ⁶: Let \((X, \tau, \mathcal{C})\) be a topological convex structure. Then collection of all closed sets in \((X, \tau)\) are subset of \(\mathcal{C}\).

**Definition 2.7** ⁷: Let \((X, \tau)\) be a topological space and let \(\mathcal{C}\) be a convexity on \(X\). Then the triplet \((X, \tau, \mathcal{C})\) is called a convex topological space \((\text{CTS in short})\).

**Definition 2.8** ⁸: Let \((X, \tau, \mathcal{C})\) be a convex topological space. A set \(P \subseteq X\) is said to be \(C\)-regular open if \(P = \text{int}(\text{co}(P))\).

**Result 2.9** ⁸: Let \(A\) be a subset of a convex topological space \((X, \tau, \mathcal{C})\). Then \(\text{int}(\text{co}(A))\) is a \(C\)-regular open set.

**Note 2.10** ⁸: In a convex topological space \((X, \tau, \mathcal{C})\) for any subset \(A\) of \(X\), the set \(\text{int}(\text{co}(A))\) is a \(C\)-regular open set. Also a subset \(B\) of \(X\) is called \(C\)-regular closed set if its complement is \(C\)-regular open set.

**Definition 2.11** ⁸: \([8]\) Let \((X, \tau, \mathcal{C})\) be a convex topological space. Let \(S\) be a subset of \(X\) and \(x \in X\).

(a) \(x\) is called \(\delta_x - \mathcal{C}\) cluster point of \(S\) if \(S \cap \text{int}(\text{co}(U)) \neq \emptyset\), for each open nbd. \(U\) of \(x\).

(b) The family of all \(\delta_x - \mathcal{C}\) cluster points of \(S\) is called the \(\delta_x - \mathcal{C}\) closure of \(S\) and is denoted by \([S]_{\delta_x}\).

(c) A subset \(S\) of \(X\) is called \(\delta_x - \mathcal{C}\) closed if \([S]_{\delta_x} = S\). The complement of a \(\delta_x - \mathcal{C}\) closed set is said to be a \(\delta_x - \mathcal{C}\) open set.

**Definition 2.12** ⁸: Let \((X, \tau, \mathcal{C}_1)\) and \((Y, \sigma, \mathcal{C}_2)\) be two convex topological spaces. A function \(f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)\) is said to be \(\delta_x - \mathcal{C}\) continuous if for each \(x \in X\) and each open nbd. \(V\) of \(f(x)\), there exists an open nbd. \(U\) of \(x\) such that \(f\left(\text{int}(\text{co}(U))\right) \subseteq \text{int}(\text{co}(V))\).

**Result 2.13** ⁸: A function \(f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)\) is \(\delta_x - \mathcal{C}\) continuous iff for each \(x \in X\) and each \(C\)-regular open set \(V\) containing \(f(x)\), there exists a \(C\)-regular open set \(U\) containing \(x\) such that \(f(U) \subseteq V\).
3. COMPARISON OF DIFFERENT TYPES OF CONTINUOUS FUNCTIONS:

Definition 3.1: Let \((X, \tau, C_1)\) and \((Y, \sigma, C_2)\) be two convex topological spaces. A function \(f : (X, \tau, C_1) \rightarrow (Y, \sigma, C_2)\) is said to be strongly \(\theta_* - C\) continuous, \(\theta_* - C\) continuous, regular \(C\)-continuous if for each \(x \in X\) and each open nbd. \(V\) of \(f(x)\), there exists an open nbd. \(U\) of \(x\) such that \(f(co(U)) \subseteq V\), \(f(co(U)) \subseteq co(V)\), \(f(U) \subseteq int(co(V))\) respectively.

Remark 3.2: For any convex topological space we have \(P \subseteq co(P)\). This shows that strongly \(\theta_* - C\) continuous \(\Rightarrow \theta_* - C\) continuous.

The following example shows that the converse of the above implication may not be true in general.

Example 3.3: Let us consider the function \(f : (X, \tau, C_1) \rightarrow (X, \sigma, C_2)\) where \(X = \{a, b, c\}\), \(\tau = \{\emptyset, X\}\), \(C_1 = \{\emptyset, X\}\), \(\sigma = \{\emptyset, X\}\), \(C_2 = \{\emptyset, X\}\) and \(f\) is the identity mapping \(I_X\) on \(X\).

Here \(f\) is \(\theta_* - C\) continuous on \(X\). Also for \(a \in X\) if we consider the open nbd. \(\{a\}\) of \(f(a)\), then there is no \(U \in \tau\) such that \(f(co(U)) \subseteq \{a\}\). So \(f\) is not strongly \(\theta_* - C\) continuous on \(X\).

Note 3.4: We now show that \(\theta_* - C\) continuous and regular \(C\)-continuous are two independent concepts which follows from the next two examples.

Example 3.5: Let us consider the function \(f : (X, \tau, C_1) \rightarrow (X, \sigma, C_2)\) where \(X = \{a, b, c\}\), \(\tau = \{\emptyset, X\}, \{a, b\}\), \(C_1 = \{\emptyset, X\}, \{a, b\}\), \(\sigma = \{\emptyset, X\}\), \(C_2 = \{\emptyset, X\}\) and \(f\) is the identity mapping \(I_X\) on \(X\).

Here \(f\) is \(\theta_* - C\) continuous but not regular \(C\)-continuous on \(X\).

Example 3.6: Let us consider the function \(f : (X, \tau, C_1) \rightarrow (X, \sigma, C_2)\) where \(X = \{a, b, c\}\), \(\tau = \{\emptyset, X\}, \{a\}, \{b\}\), \(C_1 = \{\emptyset, X\}\), \(\sigma = \{\emptyset, X\}\), \(C_2 = \{\emptyset, X\}\) and \(f\) is the identity mapping \(I_X\) on \(X\).

Here \(f\) is regular \(C\)-continuous but not \(\theta_* - C\) continuous on \(X\).

Theorem 3.7: Let a function \(f : (X, \tau, C_1) \rightarrow (Y, \sigma, C_2)\) be \(\theta_* - C\) continuous and open. Then \(f\) is regular \(C\)-continuous function.

Proof: Let \(x \in X\) and \(V\) be an open nbd. of \(f(x)\). Since \(f\) is \(\theta_* - C\) continuous, there exists an open nbd. \(U\) of \(x\) such that \(f(co(U)) \subseteq co(V)\). Thus \(f(U) \subseteq f(co(U)) \subseteq co(V)\).

Now \(U\) is an open set and \(f\) is open mapping. Thus \(f(U)\) is an open set in \(Y\) which is contained in \(co(V)\). So \(f(U) \subseteq int(co(V))\). Consequently \(f\) is regular \(C\)-continuous function.
Theorem 3.8: (a) If a function \( f : (X, \tau, C_1) \to (Y, \sigma, C_2) \) is strongly \( \theta_* - \mathcal{C} \) continuous and \( g : (Y, \sigma, C_2) \to (Z, \gamma, C_3) \) is regular \( \mathcal{C} \) continuous, then \( g \circ f : (X, \tau, C_1) \to (Z, \gamma, C_3) \) is \( \delta_* - \mathcal{C} \) continuous.

(b) The following implications hold:

- strongly \( \theta_* - \mathcal{C} \) continuous \( \Rightarrow \delta_* - \mathcal{C} \) continuous \( \Rightarrow \) regular \( \mathcal{C} \) continuous.

Proof: (a) Let \( x \in X \) and \( W \) be any open set containing \((g \circ f)(x)\). Since \( g \) is regular \( \mathcal{C} \) continuous, there exists an open nbd. \( V \) of \( f(x) \) in \( Y \) such that \( g(V) \subseteq \text{int}(\text{co}(W)) \). Again since \( f \) is strongly \( \theta_* - \mathcal{C} \) continuous, there exists an open nbd. \( U \) of \( x \) in \( X \) such that \( f(\text{co}(U)) \subseteq V \). Now \( f(\text{int}(\text{co}(U))) \subseteq f(\text{co}(U)) \subseteq V \Rightarrow g(f(\text{int}(\text{co}(U)))) \subseteq g(f(\text{co}(U))) \subseteq \text{int}(\text{co}(W)) \Rightarrow (g \circ f)(\text{int}(\text{co}(U))) \subseteq \text{int}(\text{co}(W)). \) This shows that \( g \circ f \) is \( \delta_* - \mathcal{C} \) continuous.

(b) Let \( f \) be strongly \( \theta_* - \mathcal{C} \) continuous. Also let \( x \in X \) and \( V \) be any open nbd. of \( f(x) \). Then there exists an open nbd. \( U \) of \( x \) in \( X \) such that \( f(\text{co}(U)) \subseteq V \Rightarrow f(\text{int}(\text{co}(U))) \subseteq f(\text{co}(U)) \subseteq V = \text{int}(V) \subseteq \text{int}(\text{co}(V)) \). Hence \( f \) is \( \delta_* - \mathcal{C} \) continuous.

Again let \( f \) be \( \delta_* - \mathcal{C} \) continuous. Also let \( x \in X \) and \( V \) be any open nbd. of \( f(x) \) in \( Y \). Then there exists an open nbd. \( U \) of \( x \) in \( X \) such that \( f(\text{int}(\text{co}(U))) \subseteq \text{int}(\text{co}(V)) \). Let \( W = \text{int}(\text{co}(U)) \). Then \( W \) is open nbd. of \( x \) such that \( f(W) \subseteq \text{int}(\text{co}(V)) \). Thus \( f \) is regular \( \mathcal{C} \) continuous.

Note 3.9: The following examples show that none of these implications in the above theorem is reversible.

Example 3.10: Let us consider the function \( f : (X, \tau, C_1) \to (X, \sigma, C_2) \) where \( X = \{a, b\} \), \( \tau = \{\emptyset, X, \{a\}\} \), \( C_1 = \{\emptyset, X\} \), \( C_2 = \{\emptyset, X\} \) and \( f \) is the identity mapping \( I_X \) on \( X \).

Here \( f \) is \( \delta_* - \mathcal{C} \) continuous but not strongly \( \theta_* - \mathcal{C} \) continuous on \( X \).

Example 3.11: Let us consider the function \( f : (X, \tau, C_1) \to (X, \sigma, C_2) \) where \( X = \{a, b, c\} \), \( \tau = \{\emptyset, X, \{a\}, \{b\}\} \), \( C_1 = \{\emptyset, X\} \), \( \sigma = \{\emptyset, X, \{a\}\} \), \( C_2 = \{\emptyset, X, \{a\}\} \) and \( f \) is the identity mapping \( I_X \) on \( X \).

Here \( f \) is regular \( \mathcal{C} \) continuous but not \( \delta_* - \mathcal{C} \) continuous on \( X \).

Definition 3.12: A convex topological space \((X, \tau, C)\) is said to be an semi \( \mathcal{C} \) regular space if for each \( x \in X \) and each open nbd. \( V \) of \( x \) there exists an open nbd. \( U \) of \( x \) such that \( x \in U \subseteq \text{int}(\text{co}(U)) \subseteq V \).
**Theorem 3.13:** For a function \( f : (X, \tau, C_1) \rightarrow (Y, \sigma, C_2) \) the following properties are true:

(a) If \( Y \) is a semi-\( C \)-regular space and \( f \) is \( \delta_* - C \) continuous, then \( f \) is continuous.

(b) If \( X \) is a semi-\( C \)-regular space and \( f \) is regular-\( C \)-continuous, then \( f \) is \( \delta_* - C \) continuous.

**Proof:** (a) Let \( Y \) be a semi-\( C \)-regular space and \( x \in X \). Then for each open nbd. \( V \) of \( f(x) \), there exists an open nbd. \( W \) of \( f(x) \) such that \( f(x) \in W \subseteq \text{int}(\text{co}(W)) \subseteq V \). Since \( f \) is \( \delta_* - C \) continuous, there exists an open nbd. \( U \) of \( x \) such that \( f\left(\text{int}(\text{co}(U))\right) \subseteq \text{int}(\text{co}(W)) \subseteq V \) i.e., \( f(U) \subseteq V \). Hence \( f \) is continuous.

(b) Let \( x \in X \) and \( V \) be an open nbd. of \( f(x) \). Since \( f \) is regular-\( C \)-continuous, there exists an open nbd. \( U \) of \( x \) such that \( f(U) \subseteq \text{int}(\text{co}(V)) \). Again since \( X \) is a semi-\( C \)-regular space there exists an open nbd. \( W \) of \( x \) such that \( \text{int}(\text{co}(W)) \subseteq U \). Thus \( f\left(\text{int}(\text{co}(W))\right) \subseteq f(U) \subseteq \text{int}(\text{co}(V)) \). Hence \( f \) is \( \delta_* - C \) continuous.

**Corollary 3.14:** If \( (X, \tau, C_1) \) and \( (Y, \sigma, C_2) \) are semi-\( C \)-regular spaces, then the concepts on a function \( f : (X, \tau, C_1) \rightarrow (Y, \sigma, C_2) \), \( \delta_* - C \) continuity, continuity, regular-\( C \)-continuity are equivalent.

**Definition 3.15:** A CTS \( (X, \tau, C) \) is said to be an almost-\( C \)-regular if for each \( C \)-regular closed set \( F \) and each \( x \notin F \), there exist disjoint open sets \( U \) and \( V \) such that \( x \in U \) and \( F \subseteq V \).

**Theorem 3.16:** Let \( (X, \tau, C) \) be an almost-\( C \)-regular space where \( \tau \) is compatible with \( C \). Then for each \( x \in X \) and each \( C \)-regular open nbd. \( V \) of \( x \), there exists a \( C \)-regular open nbd. \( W \) of \( x \) such that \( x \in W \subseteq \text{co}(W) \subseteq V \).

**Proof:** Let \( x \in X \) and \( V \) be a \( C \)-regular open set containing \( x \). Then \( x \notin X \setminus V \) and \( X \setminus V \) is a \( C \)-regular closed set. Thus there exist disjoint open sets \( U_1, U_2 \) such that \( x \in U_1 \) and \( X \setminus V \subseteq U_2 \). Now \( U_1 \cap U_2 = \emptyset \Rightarrow \text{cl}(U_1) \cap U_2 = \emptyset \Rightarrow \text{co}(U_1) \cap U_2 = \emptyset \). Since \( \tau \) is compatible with \( C \) \( \Rightarrow \text{co}(U_1) \subseteq X \setminus U_2 \subseteq V \Rightarrow \text{int}(\text{co}(U_1)) \subseteq V \). Let \( W = \text{int}(\text{co}(U_1)) \). Then \( x \in W \) and \( W \) is a \( C \)-regular open set. Also \( W = \text{int}(\text{co}(U_1)) \subseteq \text{co}(U_1) \Rightarrow \text{co}(W) \subseteq \text{co}(U_1) \subseteq V \Rightarrow x \in W \subseteq \text{co}(W) \subseteq V \).

**Theorem 3.17:** For a function \( f : (X, \tau, C_1) \rightarrow (Y, \sigma, C_2) \) the following hold:

1. If \( Y \) is almost-\( C \)-regular space where \( \sigma \) is compatible with \( C_2 \) and \( f \) is \( \theta_* - C \) continuous, then \( f \) is \( \delta_* - C \) continuous.
2. If \( X \) is almost-\( C \)-regular space where \( \tau \) is compatible with \( C_1 \), \( Y \) is semi-\( C \)-regular space and \( f \) is \( \delta_* - C \) continuous, then \( f \) is strongly \( \theta_* - C \) continuous.
Proof: 1) Let $x \in X$ and $V$ be a $\mathcal{C}$-regular open nbd. of $f(x)$. Then $Y$ being almost $\mathcal{C}$-regular space, there exist a $\mathcal{C}$-regular open nbd. $U$ of $f(x)$ such that $f(x) \in U \subseteq \text{co}(U) \subseteq V$. Since $f$ is $\theta_* - \mathcal{C}$ continuous, there exists an open nbd. $W$ of $x$ such that $f(\text{co}(W)) \subseteq \text{co}(U)$. Thus
\[
\big(\text{int}(\text{co}(W))\big) \subseteq f(\text{co}(W)) \subseteq \text{co}(U) \subseteq V.
\] Hence $f$ is $\delta_* - \mathcal{C}$ continuous.

2) Let $x \in X$ and $V$ be an open nbd. of $f(x)$. Since $Y$ is semi $\mathcal{C}$-regular space, there exists an open nbd. $U$ of $f(x)$ such that $f(x) \in U \subseteq \text{int}(\text{co}(U)) \subseteq V$. Again by the $\delta_* - \mathcal{C}$ continuity of $f$, there exists an open nbd. $W$ of $x$ such that $f\left(\text{int}(\text{co}(W))\right) \subseteq \text{int}(\text{co}(U))$. Now $\text{int}(\text{co}(W))$ is a $\mathcal{C}$-regular open set in $X$ which is almost $\mathcal{C}$-regular. So there is $\mathcal{C}$-regular open nbd. $P$ of $x$ such that $x \in P \subseteq \text{co}(P) \subseteq \text{int}(\text{co}(W))$. This implies that $f(\text{co}(P)) \subseteq f\left(\text{int}(\text{co}(W))\right) \subseteq \text{int}(\text{co}(U)) \subseteq V$. Thus $f$ is strongly $\theta_* - \mathcal{C}$ continuous.

**Definition 3.18**: A function $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ is called $\mathcal{C}$-regular open if for each $\mathcal{C}$-regular open set $U$ of $X$, $f(U)$ is open in $Y$.

**Theorem 3.19**: Let a function $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ be $\theta_* - \mathcal{C}$ continuous and $\mathcal{C}$-regular open. Then $f$ is $\delta_* - \mathcal{C}$ continuous function.

Proof: Let $x \in X$ and $V$ be an open nbd. of $f(x)$. Since $f$ is $\theta_* - \mathcal{C}$ continuous, there exists an open nbd. $U$ of $x$ such that $f(\text{co}(U)) \subseteq \text{co}(V)$. Thus $f\left(\text{int}(\text{co}(U))\right) \subseteq f(\text{co}(U)) \subseteq \text{co}(V)$. Now $\text{int}(\text{co}(U))$ is a $\mathcal{C}$-regular open set and $f$ is $\mathcal{C}$-regular open mapping. Thus $f\left(\text{int}(\text{co}(U))\right)$ is an open set in $Y$ which is contained in $\text{co}(V)$. So $f\left(\text{int}(\text{co}(U))\right) \subseteq \text{int}(\text{co}(V))$. Consequently $f$ is $\delta_* - \mathcal{C}$ continuous function.

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