Research article Available online www.ijsrr.org ISSN: 2279–0543

International Journal of Scientific Research and Reviews

Certain Subclass of Analytic Univalent Functions Using q - Differential Operator

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ABSTRACT

In this paper, we define a new subclass of analytic univalent function using q - differential operator, which generalizes Ruschewayh differential operators. Coefficient inequalities, Subordination, extreme points and integral means inequalities results are obtained.

KEYWORDS: Univalent, subordinating factor sequence, integral means, Hadamard product, q-derivative operator, subordination.

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INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let us denote $\mathrm{U}=\{z\in\mathbb{C}\setminus|z|<1\}$ and let A denote the class of functions of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots,\qquad a_n\geq 0,\tag{1.1}$$

which are analytic in the open disc U.

Mohammed and Darus[1] studied approximation and geometric properties of the $q-$operators in some subclasses of analytic functions in compact disk. Recently, Purohit and Raina[2,3] have used the fractional $q-$calculus operators in investigating certain classes of functions which are analytic in the open disk. Also Purohit[4] studied these $q-$operators, defined by using the convolution of normalized analytic functions and $q-$hypergeometric functions.

The $q-$derivative operator of a function f is defined by

$$D_q f(z)=\frac{f(z)-f(qz)}{(1-q)z}\quad (z\neq 0)\tag{1.2}$$

and $(D_q f)(0)=f'(0)$, provided that the function f is differentiable at 0. We note that $D_q f(z)\to f'(z)$ as $q\to 1^-$.

Also, from [1.2], we have $D\_\{q\}\ f(z)=1++\sum_{n=2}^{\infty}[n]\ a_n\ z^n$, where

$$[n]=\frac{1-q^n}{1-q}.\tag{1.3}$$

The Hadamard product of two functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ and $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ is given by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_n z^n.\tag{1.4}$$

Recently, Kanas and Raducanu [5], defined and investigated Ruschewayh $q-$differential operator as follows:

For $f\in\mathrm{A}$, generalized Ruschewayh $q-$differential operator is defined by

$$R_q^{\lambda} f(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma_q(n+\lambda)}{(n-1)!\Gamma_q(1+\lambda)}a_nz^n,\qquad \left(z\in\mathrm{U}\right).\tag{1.5}$$

Here $R_q^0 f(z)=f(z), R_q^1 f(z)=zD_q f(z)$ and $R_q^1 f(z)=\dfrac{zD_q^m(z^{m-1}f(z))}{[m]!}$.

It can be seen that if we let $q\to 1^-$, then $R_q^{\lambda} f\left(z\right)$ reduces to the well-known Ruschewayh differential operator[6]. Using the operator $R_q^{\lambda} f\left(z\right)$ and (1.2), we obtain

$$D_q\left(R_q^{\lambda} f(z)\right)= 1+ \sum_{n=2}^{\infty} [\mathrm{n}]\ \psi_q(n,\lambda)\ a_n z^{n-1} \tag{1.6}$$

where

$$\psi_q(n,\lambda)=\frac{\Gamma_q(n+\lambda)}{(n-1)!\Gamma_q(1+\lambda)}. \tag{1.7}$$

Using the generalized Ruschewayh $q-$differential operator, we define the following class $\mathrm{S}_q(n,\lambda,\mu)$.

Definition 1.1: Let $\mathrm{S}_q(n,\lambda,\mu)$ be the class of functions $f\in \mathrm{A}$ satisfying

$$Re\left\{\frac{D_q\left(R_q^{\lambda} f\left(z\right)\right)}{D_q\left(R_q^{\mu} f\left(z\right)\right)}\right\}>\alpha \tag{1.8}$$

for some $0\le\alpha<1,\lambda\in\square_{-1}^{0}=\square_{-1}-\{0\},\mu\in\square_{-1}$.

Definition 1.2: (Subordination) Given two functions $f\left(z\right)$ and $g\left(z\right)$, which are analytic in U. Then we say that the function $f(z)$ is subordinate to $g(z)$ in U, if there exists an analytic function $w(z)$ in U such that $w(0)=0, |w(z)|<1 (z\in\mathrm{U})$ such that $f(z)=g(w(z))$, denoted by $f(z)\prec g(z)$.

Definition 1.3: (Subordinating Factor Sequence) A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating sequence if, whenever $f(z)=\sum_{n=1}^{\infty} a_n z^n$, $a_1=1$ is regular, univalent and convex in U, we have

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z), \qquad (z\in\mathrm{U}). \tag{1.9}$$

Motivated by the concept introduced by Serap Bulut[7], Selvaraj[8], in this paper, we obtain coefficient bounds, extreme points and integral means inequalities for the above said function class.

Let T denote the subclass of $f\in\mathrm{A}$ consisting of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n \tag{1.10}$$

COEFFICIENT INEQUALITIES

Theorem 2.1: Let $f(z)\in\mathrm{A}$ of the form [1.1]. If the inequality

$$\sum_{n=2}^{\infty}\mathrm{B}_n(\lambda,\mu,\alpha)|a_n|\le 2(1-\alpha), \qquad \left(z\in\mathrm{U}\right), \tag{2.1}$$

holds true for some $0\le\alpha<1,\lambda\in\square_{-1}^{0},\mu\in\square_{-1}$, where

$$\mathrm{B}_n(\lambda,\mu,\alpha)=[n]\Big[\left|\psi_q(n,\lambda)-(1+\alpha)\psi_q(n,\mu)\right|+\left|\psi_q(n,\lambda)+(1-\alpha)\psi_q(n,\mu)\right|\Big] \tag{2.2}$$

and $\psi_q(n,\lambda)$ is given by (1.7) then $f\in \mathrm{S}_q(n,\lambda,\mu)$.

Proof: Suppose that the inequality (2.1) holds. Then for $z\in \mathrm{U}$, we define the function F by

$$F(z)=\frac{D_q\left(R_q^{\lambda} f(z)\right)}{D_q\left(R_q^{\mu} f(z)\right)}-\alpha. \tag{2.3}$$

It is sufficient to show that

$$\left|\frac{F(z)-1}{F(z)+1}\right|<1, \qquad (z\in \mathrm{U}). \tag{2.4}$$

Now,

$$\left|\frac{F(z)-1}{F(z)+1}\right|=\left|\frac{D_q\left(R_q^{\lambda} f(z)\right)-(1+\alpha)D_q\left(R_q^{\mu} f(z)\right)}{D_q\left(R_q^{\lambda} f(z)\right)+(1-\alpha)D_q\left(R_q^{\mu} f(z)\right)}\right|$$

$$=\left|\frac{\alpha-\sum_{n=2}^{\infty}[n]\left[\psi_q(n,\lambda)-(1+\alpha)\psi_q(n,\mu)\right]a_n z^{n-1}}{(2-\alpha)+\sum_{n=2}^{\infty}[n]\left[\psi_q(n,\lambda)-(1+\alpha)\psi_q(n,\mu)\right]a_n z^{n-1}}\right|$$

$$\le\frac{\alpha-\sum_{n=2}^{\infty}[n]\left[\left|\psi_q(n,\lambda)-(1+\alpha)\psi_q(n,\mu)\right|\right]|a_n||z|^{n-1}}{(2-\alpha)-\sum_{n=2}^{\infty}[n]\left[\left|\psi_q(n,\lambda)-(1+\alpha)\psi_q(n,\mu)\right|\right]|a_n||z|^{n-1}}$$

$$<\frac{\alpha-\sum_{n=2}^{\infty}[n]\left[\left|\psi_q(n,\lambda)-(1+\alpha)\psi_q(n,\mu)\right|\right]|a_n|}{(2-\alpha)-\sum_{n=2}^{\infty}[n]\left[\left|\psi_q(n,\lambda)-(1+\alpha)\psi_q(n,\mu)\right|\right]|a_n|} \quad < \quad 1$$

Therefore, $f\in \mathrm{S}_q(n,\lambda,\mu)$.

INTEGRAL MEANS INEQUALITIES

Lemma 1: [Selvaraj et al.[9]] If the functions f and g are analytic in U with $f(z)\prec g(z)$, then for $\beta>0$ *and* $z=re^{i\theta}(0<r<1)$,

$$\int_0^{2\pi}\left|f(z)\right|^{\beta}d\theta\le\int_0^{2\pi}\left|g(z)\right|^{\beta}d\theta. \tag{3.1}$$

Silverman[10] found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T and applied this function to resolve his integral means inequality, conjectured and settled in[11], that

$$\int_0^{2\pi} |f(z)|^\beta d\theta \le \int_0^{2\pi} |f_2(z)|^\beta d\theta. \tag{3.2}$$

for all $f \in \mathrm{T}$. In[12], Silverman also proved his conjecture for the subclasses $\mathrm{T}^*(\alpha)$ and $\mathrm{K}(\alpha)$ of T.

Theorem 3.1: Suppose $f \in \mathrm{S}_q(n,\lambda,\mu)$, $\beta > 0$, $0 \le \alpha < 1$ and $f_2(z)$ is defined by

$$f_2(z) = z - \frac{2(1-\alpha)}{\mathrm{B}_2(\lambda,\mu,\alpha)} z^2, \text{ where}$$

$$\mathrm{B}_2(\lambda,\mu,\alpha) = \left[\left|\psi_q(2,\lambda) - (1+\alpha)\psi_q(2,\mu)\right| + \left|\psi_q(2,\lambda) + (1-\alpha)\psi_q(2,\mu)\right|\right] \tag{3.3}$$

and $$\psi_q(2,\lambda) = [2]\frac{\Gamma_q(2+\lambda)}{(2-1)!\Gamma_q(1+\lambda)}. \tag{3.4}$$

Then for $z = re^{i\theta}(0 < r < 1)$, we have

$$\int_0^{2\pi} |f(z)|^\beta d\theta \le \int_0^{2\pi} |f_2(z)|^\beta d\theta. \tag{3.5}$$

Proof: Using (1.10) and (3.5), it is enough to prove that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} |a_n| z^{n-1}\right|^\beta d\theta \le \int_0^{2\pi} \left|1 - \frac{2(1-\alpha)}{\mathrm{B}_2(\lambda,\mu,\alpha)} z\right|^\beta d\theta. \tag{3.6}$$

By Lemma 1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{2(1-\alpha)}{\mathrm{B}_2(\lambda,\mu,\alpha)} z.$$

Setting $$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{2(1-\alpha)}{\mathrm{B}_2(\lambda,\mu,\alpha)} w(z),$$

and using (2.1), we obtain $w(z)$ is analytic in U, $w(0) = 0$, and

$$|w(z)| = \left|\sum_{n=2}^{\infty} \frac{\mathrm{B}_2(\lambda,\mu,\alpha)}{2(1-\alpha)} |a_n| z^{n-1}\right| \le |z| \sum_{n=2}^{\infty} \frac{\mathrm{B}_n(\lambda,\mu,\alpha)}{2(1-\alpha)} |a_n| \le |z|,$$

where $\mathrm{B}_n(\lambda,\mu,\alpha)$ is given by (2.2). This completes the proof of the theorem.

SUBORDINATION RESULTS

Motivated by the concept introduced by Frasin[12] and Singh[13], we obtain subordination results for the function class $\mathrm{S}_q(n,\lambda,\mu)$.

Lemma 2: The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if

$$Re\left(1+2\sum_{n=1}^{\infty} b_n z^n\right)>0, \qquad (z\in \mathrm{U}). \tag{4.1}$$

Theorem 4.1: Let $f\in \mathrm{S}_q(n,\lambda,\mu)$ and $g(z)$ be any function in the usual class of convex functions K, then

$$\frac{\mathrm{B}_2(\lambda,\mu,\alpha)}{2\left[2(1-\alpha)+\mathrm{B}_2(\lambda,\mu,\alpha)\right]}(f*g)(z)\prec g(z) \qquad (z\in \mathrm{U}) \tag{4.2}$$

where $0\le\alpha<1$ with $\psi_q(n,\lambda)$ given by (3.4) and

$$Re\left(f(z)\right)>-\frac{\left[2(1-\alpha)+\mathrm{B}_2(\lambda,\mu,\alpha)\right]}{\mathrm{B}_2(\lambda,\mu,\alpha)} \qquad (z\in \mathrm{U}). \tag{4.3}$$

Proof: Let $f\in \mathrm{S}_q(n,\lambda,\mu)$ and suppose that $g(z)=z+\sum_{n=2}^{\infty} b_n z^n\in \mathrm{K}$.

Then, for $f\in \mathrm{A}$ given by (1.1), we have

$$\frac{\mathrm{B}_2(\lambda,\mu,\alpha)}{2\left[2(1-\alpha)+\mathrm{B}_2(\lambda,\mu,\alpha)\right]}(f*g)(z)=\frac{\mathrm{B}_2(\lambda,\mu,\alpha)}{2\left[2(1-\alpha)+\mathrm{B}_2(\lambda,\mu,\alpha)\right]}\left(z+\sum_{n=2}^{\infty} a_n b_n z^n\right). \tag{4.4}$$

Thus, by Definition 3, the subordination result holds true if $\left(\frac{\mathrm{B}_2(\lambda,\mu,\alpha)}{2\left[2(1-\alpha)+\mathrm{B}_2(\lambda,\mu,\alpha)\right]}\right)_{n=1}^{\infty}$

is a subordinating factor sequence, with $a_1=1$. In view of Lemma 2, this is equivalent to the following inequality

$$Re\left(1+\sum_{n=1}^{\infty}\frac{\mathrm{B}_2(\lambda,\mu,\alpha)}{\left[2(1-\alpha)+\mathrm{B}_2(\lambda,\mu,\alpha)\right]}a_n z^n\right)>0, \qquad (z\in \mathrm{U}). \tag{4.5}$$

Now, for $|z|=r<1$, we have

$$\begin{aligned} &Re\left(1+\frac{\mathrm{B}_2(\lambda,\mu,\alpha)}{\left[2(1-\alpha)+\mathrm{B}_2(\lambda,\mu,\alpha)\right]}\sum_{n=1}^{\infty} a_n z^n\right)\\ &=Re\left(1+\frac{\mathrm{B}_2(\lambda,\mu,\alpha)}{\left[2(1-\alpha)+\mathrm{B}_2(\lambda,\mu,\alpha)\right]}z+\frac{\sum_{n=1}^{\infty}\mathrm{B}_2(\lambda,\mu,\alpha)a_n z^n}{\left[2(1-\alpha)+\mathrm{B}_2(\lambda,\mu,\alpha)\right]}\right) \end{aligned} \tag{4.6}$$

$$\begin{aligned}&\geq 1-\frac{\mathrm{B}_2(\lambda,\mu,\alpha)}{\left[2(1-\alpha)+\mathrm{B}_2(\lambda,\mu,\alpha)\right]}r-\frac{\sum_{n=1}^{\infty}\mathrm{B}_2(\lambda,\mu,\alpha)a_n r^n}{\left[2(1-\alpha)+\mathrm{B}_2(\lambda,\mu,\alpha)\right]}\\&\geq 1-\frac{\mathrm{B}_2(\lambda,\mu,\alpha)}{\left[2(1-\alpha)+\mathrm{B}_2(\lambda,\mu,\alpha)\right]}r-\frac{2(1-\alpha)}{\left[2(1-\alpha)+\mathrm{B}_2(\lambda,\mu,\alpha)\right]}r\\&=1-r>0, \qquad (|z|=r<1)\end{aligned}$$

where we have also made use of the assertion (2.1) of Theorem 2.1. This evidently proves the inequality (4.5) and hence also the subordination result (4.4) asserted by Theorem 4.1. The inequality (4.3) asserted by Theorem 4.1 would follow from (4.2) upon setting

$$g(z)=\frac{z}{1-z}=\sum_{j=1}^{\infty}z^j\in\mathrm{K}.$$

Finally, we consider the function $q(z)$ is given by

$$q(z)=z-\frac{2(1-\alpha)}{\mathrm{B}_2(\lambda,\mu,\alpha)}z^2 \qquad (4.7)$$

and $\psi_q(2,\lambda)$ is given by (3.4). Clearly $q\in\mathrm{S}_q(n,\lambda,\mu)$.

For this function (4.2) becomes

$$\frac{\mathrm{B}_2(\lambda,\mu,\alpha)}{2\left[2(1-\alpha)+\mathrm{B}_2(\lambda,\mu,\alpha)\right]}q(z)\prec\frac{z}{1-z}.$$

Moreover, it can easily be verified for the function $q(z)$ given by (4.7) that

$$min\left\{Re\left(\frac{\mathrm{B}_2(\lambda,\mu,\alpha)}{2\left[2(1-\alpha)+\mathrm{B}_2(\lambda,\mu,\alpha)\right]}q(z)\right)\right\}=-\frac{1}{2} \qquad (z\in\mathrm{U}),$$

which evidently completes the proof of Theorem 4.1.

EXTREME POINTS

Theorem 5.1: Let $f_1(z)=z$ and

$$f_k(z)=z+\frac{2(1-\alpha)}{\mathrm{B}_k(\lambda,\mu,\alpha)}z^k \qquad (k=2,3,\ldots) \qquad (5.1)$$

where $\mathrm{B}_k(\lambda,\mu,\alpha)$ given by (2.2). Then $f\in\mathrm{S}_q(n,\lambda,\mu)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=1}^{\infty}\eta_k f_k(z), \qquad (5.2)$$

where $\eta_k\geq 0$ and $\sum_{k=1}^{\infty}\eta_k=1$.

Proof: Assume that (5.2) holds true. Then

$$
\begin{aligned}
f(z) &= \eta_1 f_1(z) + \sum_{k=2}^{\infty} \eta_k f_k(z), \\
&= \eta_1 z + \sum_{k=2}^{\infty} \eta_k \left(z + \frac{2(1-\alpha)}{\mathrm{B}_k(\lambda,\mu,\alpha)} z^k \right) \qquad (5.3) \\
&= \left(\sum_{k=1}^{\infty} \eta_k \right) z + \sum_{k=2}^{\infty} \left(\eta_k \frac{2(1-\alpha)}{\mathrm{B}_k(\lambda,\mu,\alpha)} z^k \right) = z + \sum_{k=2}^{\infty} \left(\eta_k \frac{2(1-\alpha)}{\mathrm{B}_k(\lambda,\mu,\alpha)} z^k \right).
\end{aligned}
$$

Thus

$$
\sum_{k=2}^{\infty} \eta_k \frac{2(1-\alpha)}{\mathrm{B}_k(\lambda,\mu,\alpha)} \mathrm{B}_k(\lambda,\mu,\alpha) = 2(1-\alpha) \sum_{k=2}^{\infty} \eta_k = 2(1-\alpha)(1-\eta_1) \le 2(1-\alpha). \qquad (5.4)
$$

Therefore, we have $f \in \mathrm{S}_q(n,\lambda,\mu)$.

Conversely, suppose that $f \in \mathrm{S}_q(n,\lambda,\mu)$. Since $a_k \le \dfrac{2(1-\alpha)}{\mathrm{B}_k(\lambda,\mu,\alpha)}, \qquad (k = 2,3,\ldots),$

we can set $\qquad \eta_k = \dfrac{\mathrm{B}_k(\lambda,\mu,\alpha)}{2(1-\alpha)} a_k, \qquad (k = 2,3,\ldots)$

and $\qquad \eta_1 = 1 - \sum_{k=2}^{\infty} \eta_k.$ Now

$$
\begin{aligned}
f(z) = z + \sum_{k=2}^{\infty} a_k z^k &= \left(\sum_{k=1}^{\infty} \eta_k \right) z + \sum_{k=2}^{\infty} \left(\eta_k \frac{2(1-\alpha)}{\mathrm{B}_k(\lambda,\mu,\alpha)} z^k \right) \\
&= \eta_1 z + \sum_{k=2}^{\infty} \eta_k \left(z + \frac{2(1-\alpha)}{\mathrm{B}_k(\lambda,\mu,\alpha)} z^k \right) \\
&= \eta_1 f_1(z) + \sum_{k=2}^{\infty} \eta_k f_k(z) = \sum_{k=1}^{\infty} \eta_k f_k(z).
\end{aligned}
$$

This completes the proof of the Theorem 5.1.

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