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Unique Common Fixed Point Theorem For Weakly Compatible Mappings In Digital Metric Space

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ABSTRACT

In this paper, we prove unique common fixed point theorem for pairs of weakly compatible mapping in digital metric space. Our results extend and improve many known results in the literature. In order to validate our establish theorem and corollaries we provide an example.

KEYWORDS: Digital metric space; common fixed point theorem; compatible mappings; weakly compatible mappings.

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1. INTRODUCTION

Rosenfield[11] become the first to take into account digital topology as a tool to study digital images. Boxer[1] produced the digital versions of the topological principles and later studied digital continuous functions. Ege and Karaca[3] set up relative and reduced Lefschetz fixed point theorem for digital images and proposed the notion of a digital metric space and proved the well-known Banach Contraction Principle for digital images. Digital metric space is one of the generalizations of metric space and digital topology. Digital topology is a developing area of general topology and functional analysis which studies feature of 2D and 3D digital image. Digital topology is the study of the topological properties of images arrays. The digital version of the topological concept was given by L. Boxer [1, 2].

Fixed point theory ends in masses of packages in mathematics, computer technological, engineering, game concept, fuzzy principle, image processing and so on. In metric areas, this theory starts with the Banach fixed-point theorem which gives a optimistic technique of locating constant factors and an crucial device for solution of some issues in mathematics and engineering and consequently has been generalized in lots of methods. A foremost shift inside the area of fixed point idea got here in 1976, when Jungck [7, 8, 9,] defined the idea of commutative and compatible maps and proved the common fixed point results for such maps. Later on, Sessa[15] gave the idea of weakly compatible, and proved results for set valued maps. Certain altercations of commutativity and compatibility can also be found in [5 7 15 16]. In this paper we establish a unique common fixed point theorem satisfying the pairs of weakly compatible mappings in the context of digital metric space. An example is given in the support of our main result.

2. DEFINITIONS AND PRELIMINIERIES

Definition 2.1. [6] For a digital metric space (X, d, ρ), if a sequence $\{x_n\} \subset X \subset \mathbb{Z}^n$ is a Cauchy sequence, there is $M \in \mathbb{N}$ such that for all $n, m > M,\ we\ have\ x_n = x_m.$

Definition 2.2. [6] A sequence $\{x_n\}$ of points of a digital metric space (X, d, ρ) converges to a limit $L \in X$ if for all $\in > 0$, there is $M \in \mathbb{N}$ such that

$$d(x_n, L) < \epsilon\ for\ all\ n > M.$$

Definition 2.3. [6] A sequence $\{x_n\}$ of points of a digital metric space (X, d, ρ) converges to a limit $L \in X$ if for all $\in > 0$, there is $M \in \mathbb{N}$ such that

$$x_n = L\ for\ all\ n > M. i.e., \qquad x_n = x_{n+1} = x_{n+2} = \cdots = L$$

Definition 2.4. [4] A digital metric space (X, d, ρ) is complete if any Cauchy sequence $\{x_n\}$ converges to a point L of (X, d, ρ).

Definition 2.5. [6] A digital metric space (X, d, ρ) is complete.

Definition 2.6. [4] Let (X, d, ρ) be a digital metric space and $T: (X, d, \rho) \to (X, d, \rho)$ be a self-map. If there exists $\lambda \in [0,1)$ such that

$$d(Tx, Ty) \leq \lambda d(x, y) \; for \; all \; x, y \in X,$$

Definition 2.7. [5] Let $X \subseteq Z^n$ and (X, d, ρ) be a digital metric space. Then there does not exist a sequence $\{x_n\}$ of distinct elements in X, such that

$$d(x_{m+1}, x_m) < d(x_m, x_{m-1}) \quad for \; m = 1,2,3, \dots$$

Proposition 2.8. [6] Every digital contraction map $T: (X, d, \rho) \to (X, d, \rho)$ is digitally continuous.

Definition 2.9. [4] Suppose that (X, d, ρ) is a digital metric space and $P, Q: X \to X$, and be two self-maps defined on X. then $P \; and \; Q$ are compatible if

$$d(PQx, Qpx) \leq d(Px, Qx) \; for \; all \; \; x \in X.$$

Definition 2.10. [4] Suppose that (X, d, ρ) is a digital metric space and $P, Q: X \to X$, and be two self-maps defined on X. then $P \; and \; Q$ are weakly compatible if

$$d(PQx, Qpx) = d(Px, Qx)$$

Whenever x is a coincidence point of $P \; and \; Q$.

Definition 2.11. [4] Two maps P and Q are said to be weakly compatible if they commute at coincidence points.

3. MAIN RESULT

Now we prove a unique common fixed point theorem for pairs of weakly compatible mappings in digital metric space.

THEOREM 3.1. Let (X, d, ρ) is a complete digital metric space, let N be a nonempty closed subset of X. Let $P, Q : N \to N$ and $G, H : N \to X$ be mappings satisfying $Q(N) \subset H(N)$ and for every $x, y \in X$,

$$\Psi\big(d(Px, Qy)\big) \leq \varphi\left(d_{G,H}(x, y)\right) - \frac{1}{2}\Psi\left(d_{G,H}(x, y)\right) - \varphi\left(d_{G,H}(x, y)\right) \qquad (1)$$

Where $\Psi: [0, \infty) \to [0, \infty)$ is a continuous function such that $\Psi(\rho) = 0$ if and only if $\rho = 0$. $\varphi: [0, \infty) \to [0, \infty)$ is a lower semi-continuous function such that $\Psi(\rho) = 0$ if and only if $\rho = 0$, and

$$d_{G,H}(x, y) = \max\left\{(Gx, Hy), (Gx, Px), (Hy, Qy), \frac{1}{2}\big((Gx, Qy) + (Hy, Px)\big)\right\} \qquad (2)$$

If one of $P(N), Q(N), G(N), HN$ is a closed subset of X, then $\{P, G\}$ and $\{Q, H\}$ have a unique point of coincidence in X. Moreover, if $\{P, G\}$ and $\{Q, H\}$ are weakly compatible, then P, Q, G and H have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $Q(N) \subset G(N)$ and $P(N) \subset H(N)$, we can define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_{2n-1} = P_{x_{2n-2}} = H_{x_{2n-1}}, \;\; y_{2n} = Q_{x_{2n-1}} = G_{x_{2n}}, \qquad n = 1,2,3,4, \dots \dots$$

Suppose that $y_{n_0} = y_{n_{0+1}}$ for some n_0. then the sequence $\{y_n\}$ is constant for $n \geq n_0$. Indeed, let $n_0 = 2k$. Then $y_{2k} = y_{2k+1}$ and it follows from (1) that

$$\Psi\big((y_{2k+1}, y_{2k+2})\big) = \Psi(Px_{2k}, Qx_{2k+1},)$$
$$\leq \Psi\left(d_{G,H}(x_{2\mathrm{k}}, x_{2\mathrm{k}+1})\right) - \varphi\left(d_{G,H}(x_{2\mathrm{k}}, x_{2\mathrm{k}+1})\right), \qquad (3)$$

Where

$$d_{G,H}(x_{2\mathrm{k}}, x_{2\mathrm{k}+1})$$
$$= max\{(y_{2k}, y_{2k+1}), (y_{2k}, Px_{2k},), (y_{2k+1}, Qx_{2k+1}), \frac{1}{2}((y_{2k}, Qx_{2k+1}) + (y_{2k+1}, Px_{2k}))\}$$
$$= max\{0,0, (y_{2k+1}, y_{2k+2}), \frac{1}{2}((y_{2k}, y_{2k+2}) + 0)\}$$
$$= max\{(y_{2k+1}, y_{2k+2}), \frac{1}{2}((y_{2k}, y_{2k+2})\}$$
$$(y_{2k+1}, y_{2k+2}).$$

By (3), we get

$$\Psi(y_{2k+1}, y_{2k+2}) \leq \Psi(y_{2k+1}, y_{2k+2}) - \varphi(y_{2k+1}, y_{2k+2}),$$

And so $\varphi(y_{2k+1}, y_{2k+2}) \leq 0$ and $y_{2k+1} = y_{2k+2}$.

Similarly, if $n_0 = 2k + 1$, then one easily obtains that $y_{2k+2} = y_{2k+3}$ and the sequence$\{y_n\}$ is constant. Therefore, $\{P, G\}$ and $\{Q, H\}$ have a point of coincidence in X.

Now, suppose that $(y_n, y_{n+1}) > 0$ for each n. we shall show that for each $n = 0,1,2,3,4, \ldots,$

$$(y_{n+1}, y_{n+2}) \leq d_{G,H}(x_\mathrm{n}, x_{\mathrm{n}+1}) = (y_n, y_{n+1}) \qquad (4)$$

Using (4), we obtain that

$$\Psi(y_{2n+1}, y_{2n+2}) = \Psi(P_{x2n}, Q_{2n+1})$$
$$\leq \Psi(d_{G,H}(x_{2\mathrm{n}}, x_{2\mathrm{n}+1})) - \varphi(d_{G,H}(x_{2\mathrm{n}}, x_{2\mathrm{n}+1})) \qquad (5)$$
$$< \Psi(d_{G,H}(x_{2\mathrm{n}}, x_{2\mathrm{n}+1})).$$

On the other hand, the control function Ψ is no decreasing. Then

$$\Psi(y_{2n+1}, y_{2n+2}) \leq \left(d_{G,H}(x_{2\mathrm{n}}, x_{2\mathrm{n}+1})\right) \qquad (6)$$

Moreover, we have

$$d_{G,H}(x_{2\mathrm{n}}, x_{2\mathrm{n}+1})$$
$$= max\left\{(y_{2n}, y_{2n+1}), (y_{2n}, Px_{2n},), (y_{2n+1}, Qx_{2n+1}), \frac{1}{2}((y_{2n}, Qx_{2n+1}) + (y_{2n+1}, Px_{2n}))\right\}$$
$$= \max\left\{(y_{2n}, y_{2n+1}), (y_{2n}, y_{2n+1}), (y_{2n+1}, y_{2n+2}), \frac{1}{2}(y_{2n}, y_{2n+2})\right\}$$
$$\leq \max\left\{(y_{2n}, y_{2n+1}), (y_{2n+1}, y_{2n+2}), \frac{1}{2}(y_{2n}, y_{2n+1}) + (y_{2n+1}, y_{2n+2})\right\}$$
$$\leq \max\{(y_{2n}, y_{2n+1}), (y_{2n+1}, y_{2n+2})\}$$

If $(y_{2n+1}, y_{2n+2}) \geq (y_{2n}, y_{2n+1})$, then by using the last inequality and (5), we have $d_{G,H}(x_{2\text{n}}, x_{2\text{n}+1}) = (y_{2n+1}, y_{2n+2})$ and (6) implies that

$$\Psi(y_{2n+1}, y_{2n+2}) = \Psi\left(d_{G,H}(Px_{2\text{n}}, Qx_{2\text{n}+1})\right)$$
$$\leq \Psi(y_{2n+1}, y_{2n+2}) - \varphi(y_{2n+1}, y_{2n+2})$$

This is only possible when $\varphi(y_{2n+1}, y_{2n+2}) = 0$. it is contradiction. Hence

$(y_{2n+1}, y_{2n+2}) \leq (y_{2n}, y_{2n+1})$, and $d_{G,H}(x_{2\text{n}}, x_{2\text{n}+1}) \leq (y_{2n}, y_{2n+1})$. in a similar way, one can obtain that

$(y_{2n+3}, y_{2n+2}) \leq d_{G,H}(x_{2\text{n}+2}, x_{2\text{n}+1}) = (y_{2n+2}, y_{2n+1})$.

So (6) holds for each $n \in N$.

It follows that the sequence $\{d(y_n, y_{n+1})\}$ is nondecreasing and the limit

$$\lim_{n\to\infty}(y_n, y_{n+1}) = \lim_{n\to\infty} d_{G,H}(x_{\text{n}}, x_{\text{n}+1})$$

exists. We denote this limit by l^*. we have $l^* \geq 0$. suppose that $l^* > 0$. Then

$$\Psi(y_{n+1}, y_{n+2}) \leq \Psi\left(d_{G,H}(x_{\text{n}}, x_{\text{n}+1})\right) - \varphi\left(d_{G,H}(x_{\text{n}}, x_{\text{n}+1})\right).$$

Passing to the (upper) limit when $n \to \infty$, we get

$$\Psi(l^*) \leq \Psi(l^*) - \lim_{n\to\infty} \inf \varphi\left(d_{G,H}(x_n, x_{n+1})\right) \leq \Psi(l^*) - \varphi(l^*), \qquad (7)$$

i.e., $\varphi(l^*) \leq 0$. using the properties of control functions, we get that $l^* = 0$, which is a contradiction. Hence we have $\lim_{n\to\infty}(y_n, y_{n+1}) = 0$. Now we show that $\{y_n\}$ is a Cauchy sequence in X. It is enough to prove that $\{y_{2n}\}$ is a Cauchy sequence. Suppose the contrary. Then, for some $\in > 0$, there exist subsequence $\{y_{2n(k)}\}$ and $\{y_{2m(k)}\}$ of $\{y_{2n}\}$ such that $n(k)$ is the smallest index satisfying

$$n(k) > m(k) \text{ and } \left(y_{n(k),}\, y_{m(k)}\right) \geq \epsilon.$$

In particular, $\left(y_{n(k)-2}, y_{m(k)}\right) < \epsilon$. Using the triangle inequality and the known relation $|d(x, z - d(x, z)| \leq d(x, z)$, we obtain that

$$\lim_{k\to\infty}(y_{2n(k)}, y_{2m(k)}) = \lim_{k\to\infty}\left(y_{2\text{n}(\text{k})}, y_{2\text{m}(\text{k})-1}\right) = \lim_{k\to\infty}\left(y_{2\text{n}(\text{k})+1}, y_{2\text{m}(\text{k})}\right)$$
$$= \lim_{k\to\infty}\left(y_{2\text{n}(\text{k})+1}, y_{2\text{m}(\text{k})-1}\right) = \epsilon. \qquad (8)$$

By using the previous limits, we get that

$$\lim_{k\to\infty} d_{G,H}\left(x_{2\text{n}(\text{k})}, x_{2\text{m}(\text{k})-1}\right) = \epsilon.$$

Indeed,

$$d_{G,H}\left(x_{2\text{n}(\text{k})}, x_{2\text{m}(\text{k})-1}\right)$$

$$\max\left\{\left(y_{2n(k)}, y_{2m(k)-1}\right), \left(y_{2n(k)}, y_{2m(k)+1}\right), \left(y_{2m(k)-1}, y_{2m(k)}\right), \frac{1}{2}\left(\left(y_{2n(k)}, y_{2m(k)}\right)\right.\right.$$

$$\left.\left. + \left(y_{2n(k)}, y_{2m(k)-1}\right)\right)\right\}$$

$$\to max\{\epsilon, 0{,}0, \frac{1}{2}(\epsilon + \epsilon)\} = \epsilon.$$

Applying (7), we obtain

$$\Psi\left(y_{2n(k)+1}, y_{2m(k)}\right) = \Psi\left(P_{x_{2n(k)}}, Q_{x_{2n(k)-1}}\right)$$

$$\leq \Psi\left(d_{G,H}\left(x_{2n(k)}, x_{2m(k)-1}\right)\right) - \varphi\left(d_{G,H}\left(x_{2n(k)}, x_{2m(k)-1}\right)\right).$$

Passing to the limit $k \to \infty$, we obtain that $\Psi(\epsilon) \leq \Psi(\epsilon) - \varphi(\epsilon)$, which is contradiction. Therefore, $\{y_n\}$ is a Cauchy sequence in the complete metric (X, d). so there exists $u \in X$ such that $\lim_{n\to\infty} y_n = u$.

To prove the uniqueness property of u, suppose that u' is another point of coincidence of G and P, that is

$$u' = Gv' = Pv' \qquad (9)$$

For some $v' \in N. By$ (4), we have

$$\Psi(u', \mathrm{u}) = \Psi(\mathrm{P}v', \mathrm{Qu}) \leq \Psi\left(d_{G,H}(v', \mathrm{u})\right) - \varphi\left(d_{G,H}(v', \mathrm{u})\right)$$

Where

$$d_{G,H}(v', \mathrm{u}) = \max\left\{(u', u), 0{,}0, \frac{1}{2}\left(d_{G,H}(v', u)\right) - \varphi\left(d_{G,H}(v', u)\right)\right\}$$

It follows from (9) that $u' = u$.

Therefore, u is the unique point of coincidence of $\{P, G\}$ and $\{Q, H\}$.

Now, if $\{P, G\}$ and $\{Q, H\}$ are weakly compatible, then by (8) and (9), we have $Pu = P(Gv) = G(Pv) = Gu = w_1$ and $Qu = Q(Hu) = H(Qu) = Hu = w_2$. by (4), we have

$$\Psi(w_1, w_2) = \Psi(\mathrm{Pu}, \mathrm{Qu}) \leq \Psi\left(d_{G,H}(\mathrm{u}, \mathrm{u})\right) - \varphi\left(d_{G,H}(\mathrm{u}, \mathrm{u})\right),$$

Where

$$d_{G,H}(\mathrm{u}, \mathrm{u}) = \max\left\{(w_1, w_2), 0{,}0, \frac{1}{2}(w_1, w_2) + (w_1, w_2)\right\}$$

It follows that $w_1 = w_2$, that is,

$$Pu = Gu = Qu = Hu. \qquad (10)$$

By (4) and (10), we have

$$\Psi(\mathrm{Pu}, \mathrm{Qu}) \leq \Psi\left(d_{G,H}(\mathrm{u}, \mathrm{u})\right) - \varphi\left(d_{G,H}(\mathrm{u}, \mathrm{u})\right),$$

Where

$$d_{G,H}(\mathrm{u},\mathrm{u}) = \max\left\{(Gv, Hu), (Gv, Pv), (Hu, Qu), \frac{1}{2}(Gv, Qu) + (Pv, Qu)\right\}$$

$$(Pv, Qu).$$

Therefore, we deduce that $Pv = Qu$, that is, $u = Qu$. It follows from (10) that

$$u = Pu = Gu = Qu = Hu.$$

Then u is the unique common fixed point of P, G, H and Q.

Example 3.1. Let (X, d, ρ) is a complete digital metric space, let $X = [4, 40]$ and d be the usual metric on X. Define $P, Q, G, H: X \to X$ as follows: $PX = 4\ for\ each\ X$;

$GX = X\ if\ x \leq 16, and$ $\quad GX = 16\ if\ 16 < x < 22,$ $\quad GX = \frac{x+18}{5}\ if\ 16 \leq x \leq$

$HX = 4\ if\ x = 4\ or\ 12\ and$ $\quad GX = \frac{X+15}{5}\ if\ x > 25;$ $\quad 25$ $\quad HX = 17 + X\ if\ 13 \leq x \leq 14$

$QX = 4\ if x < 8\ or\ x > 12,$ $\quad HX = 24 + X\ if\ 4 < x < 8,$ $\quad QX = 4 + x\ if\ 14 \leq x \leq 15.$

$$HX = 16\ if\ 13 \leq x \leq 14;$$

Then $P, Q, G\ and\ H$ satisfy all the conditions of the above theorem and have a unique common fixed point $x = 4$. being compatible mappings, all $P, Q, G\ and\ H$ are weakly compatible mappings.

Corollary. 3.2. Let $P\ and\ Q$ be weakly mappings of a complete digital metric space (X, d, ρ) into itself. Suppose $P(X) \subset Q(X)$. if there exists $\alpha \in (0,1)$ and a positive integer k such that $d\left(P^k(x), P^k(y)\right) \leq \alpha d(Q(x), Q(y))$ for all x and y in X, then P and Q have a unique common fixed point.

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