Iso-S-Closedness and Iso-S*-Closedness in L-Fuzzy Topological Spaces

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ABSTRACT

Along the line of is compactness in L-fuzzy topological spaces, we introduce iso-S-closeness and iso-S*-closeness for arbitrary L-fuzzy subsets. Further CL-iso-S-closed and CL- iso-S*-closed L-fuzzy spaces are defined and studied some of the properties and obtain some relations of these spaces with other spaces.

KEYWORDS: L-fuzzy is compactness, L-fuzzy CL- iso-S-closedness, L-fuzzy CL- iso-S*-closedness.

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INTRODUCTION

In \([0,1]\) fuzzy topological space, S-closedness and S*-closedness were defined by Coker and Malakar, but the definitions are not studied in arbitrary fuzzy sets. Later Kudri and Warner have introduced good definitions of S-closedness and S*-closedness in L-fuzzy topological spaces where L is a fuzzy lattice and have studied some of their properties along the line of compactness. In 1970, Bacon introduced the notion of isocompactness in general topology. Bhaumik and Bhattacharya introduced isocompactness in L-fuzzy topological spaces, in which every L-fuzzy closed, countably compact subspaces are L-fuzzy compact. In this paper, using the concepts of S-closedness and S*-closedness in L-fuzzy topological spaces we introduce two new concepts namely iso-S-closedness and iso-S*-closedness for arbitrary L-fuzzy subsets and study some properties of these spaces. Further we generalize these concepts as CL-iso-S-closedness and CL-iso-S*-closedness in L-fuzzy topological spaces which are the stronger form of iso-S-closedness and iso-S*-closedness.

PRIMINARIES

Throughout this paper X and Y will be non-empty ordinary sets and \(L = (\leq, \lor, \land, ')\) will denote a fuzzy lattice, i.e. a completely distributive lattice with a smallest element 0 and a largest element 1 (0 ≠ 1), and with and order reversing involution \(a \rightarrow a' (a \in L)\). An L-fuzzy subset on X is a mapping \(\lambda : X \rightarrow L\), and the family of L-fuzzy subsets on X is denoted by \(L^X\). X is called the carrier domain of each L-fuzzy subset on X.

**Definition 2.1** An element \(p\) of \(L\) is called prime if and only if \(p \neq 1\) and whenever \(a, b \in L\) with \(a \land b \leq p\) then \(a \leq p\) or \(b \leq p\). The set of all prime elements of \(L\) will be denoted by \(pr(L)\).

**Definition 2.2** An element \(\alpha\) of \(L\) is called union-irreducible or coprime if and only if whenever \(a, b \in L\) with \(\alpha \leq a \lor b\) then \(\alpha \leq a\) or \(\alpha \leq b\). The set of all nonzero union-irreducible elements of \(L\) will be denoted by \(M(L)\). It is obvious that \(p \in pr(L)\) if and only if \(p' \in M(L)\).

**Definition 2.3** Let \((X, \tau)\) be an L-fuzzy topological space and let \(\lambda \in L^X\). The L-fuzzy set \(\lambda\) is called

i) Semiopen if and only if there exists \(\beta \in \tau\) such that \(\beta \leq \lambda \leq cl(\beta)\) and semiclosed if and only if there exists a closed L-fuzzy set \(\beta\) such that \(int(\beta) \leq \lambda \leq \beta\) that is \(\lambda'\) is semiopen

ii) Pre-open if and only if \(\lambda \leq int(cl(\lambda))\) and pre-closed if and only if \(cl(int(\lambda)) \leq \lambda\) that is \(\lambda'\) is pre-open.

iii) Regularly open if and only if \(\lambda = int(cl(\lambda))\) and \(\lambda\) is regularly closed if and only if \(\lambda'\)
is regularly open i.e., \( \lambda = \text{cl}(\text{int}(\lambda)) \).

iv) Regularly semiopen\(^2\) if and only if there exist a regularly open L-fuzzy set \( \beta \) such that \( \beta \leq \lambda \leq \text{cl}(\beta) \) and \( \lambda \) is regularly semiclosed\(^2\) if and only if \( \lambda' \) is regularly semiopen.

**Definition 2.4** Let \((X, \tau)\) and \((Y, \tau')\) be two L-fuzzy topological spaces. A function \( f : (X, \tau) \to (Y, \tau') \) is called

i) Almost continuous\(^2\) if and only if \( f^{-1}(\lambda) \in \tau \) for all regularly open \( \lambda \) in \((Y, \tau')\).

ii) Almost open\(^13\) if and only if \( f(\lambda) \in \tau' \) for every regularly open \( \lambda \) in \((X, \tau)\).

iii) Weakly continuous\(^2\) if and only if \( f^{-1}(\lambda) \leq \text{int}(f^{-1}(\text{cl}(\lambda))) \) for all \( \lambda \in \tau' \).

iv) Semi-weakly continuous\(^{10}\) if and only if \( f^{-1}(\lambda) \leq \text{int}^\star(f^{-1}(\text{cl}(\lambda))) \) for all semiopen \( \lambda \in \tau' \).

v) Irresolute\(^{10}\) if and only if \( f^{-1}(\lambda) \) is semi-open in \((X, \tau)\) for every semi-open L-fuzzy set \( \lambda \) in \((Y, \tau')\).

vi) Semi-irresolute\(^{15}\) if and only if \( f^{-1}(\lambda) \) is semiclopen in \((X, \tau)\) for every semiclopen L-fuzzy set \( \lambda \) in \((Y, \tau')\).

vii) Perfect\(^{17}\) if and only if \( f \) is L-fuzzy continuous, L-fuzzy closed and for each \( y \in Y \), \( f^{-1}(y) \) is compact L-fuzzy subset in \((X, \tau)\).

**Definition 2.5** Let \((X, \tau)\) be an L-fuzzy topological space and \( \lambda \in L^X \). Then

i) The L-fuzzy subset \( \lambda \) is said to be compact\(^{12}\) if and only if for every \( p \in \text{Pr}(L) \) and every collection \((\gamma_i)_{i \in I}\) of open L-fuzzy subsets with \( (\bigvee_{i \in I} \gamma_i)(x) \not\leq p \) for all \( x \in X \) with \( \lambda(x) \geq p' \), there exists a finite subset \( F \) of \( I \) with \( (\bigvee_{i \in F} \gamma_i)(x) \not\leq p \) for all \( x \in X \) with \( \lambda(x) \geq p' \).

If \( \lambda \) is the whole space, then we say that the L-fuzzy topological space \((X, \tau)\) is compact.

ii) The L-fuzzy subset \( \lambda \) is said to be semicompact\(^{14}\) if and only if for every \( p \in \text{Pr}(L) \) and every collection \((\gamma_i)_{i \in I}\) of semiopen L-fuzzy subsets with \( (\bigvee_{i \in I} \gamma_i)(x) \not\leq p \) for all \( x \in X \) with \( \lambda(x) \geq p' \), there exists a finite subset \( F \) of \( I \) with \( (\bigvee_{i \in F} \gamma_i)(x) \not\leq p \) for all \( x \in X \) with \( \lambda(x) \geq p' \).

If \( \lambda \) is the whole space, then we say that the L-fuzzy topological space \((X, \tau)\) is semicompact.

iii) The L-fuzzy subset \( \lambda \) is said to be S-closed\(^{13}\) if and only if for every \( p \in \text{Pr}(L) \) and every collection \((\gamma_i)_{i \in I}\) of semiopen L-fuzzy subsets with \( (\bigvee_{i \in I} \gamma_i)(x) \not\leq p \) for all \( x \in X \) with \( \lambda(x) \geq p' \), there exists a finite subset \( F \) of \( I \) with \( (\bigvee_{i \in F} \text{cl}(\gamma_i))(x) \not\leq p \) for all \( x \in X \) with \( \lambda(x) \geq p' \).

If \( \lambda \) is the whole space, then we say that the L-fuzzy topological space \((X, \tau)\) is S-closed.
iv) The L-fuzzy subset $\lambda$ is said to be $S^*$-closed\(^{14}\) if and only if for every $p \in \Pr(L)$ and every collection $(\gamma_i)_{i \in I}$ of semi-open L-fuzzy subsets with $(\bigvee_{i \in I} \gamma_i)(x) \not\approx p$ for all $x \in X$ with $\lambda(x) \geq p'$, there exits a finite subset $F$ of $I$ with $(\bigvee_{i \in F} \gamma_i)(x) \not\approx p$ for all $x \in X$ with $\lambda(x) \geq p'$.

If $\lambda$ is the whole space, then we say that the L-fuzzy topological space $(X, \tau)$ is $S^*$-closed.

Other characterizations of $S$-closedness and $S^*$-closed are given in Th.2.6, Th. 2.7 and Th. 2.8.

**Theorem 2.6** Let $(X, \tau)$ be an L-fuzzy topological space and $\lambda \in L^X$. The L-fuzzy subset $\lambda$ is $S$-closed\(^{13}\) if and only if for every $p \in \Pr(L)$ and every collection $(\gamma_i)_{i \in I}$ of regularly closed L-fuzzy sets with $(\bigvee_{i \in I} \gamma_i)(x) \not\approx p$ for all $x \in X$ with $\lambda(x) \geq p'$, there is a finite subset $F$ of $I$ with $(\bigvee_{i \in F} \gamma_i)(x) \not\approx p$ for all $x \in X$ with $\lambda(x) \geq p'$.

**Theorem 2.7** Let $(X, \tau)$ be an L-fuzzy topological space and $\lambda \in L^X$. The L-fuzzy subset $\lambda$ is $S$-closed\(^{13}\) if and only if for every $p \in \Pr(L)$ and every collection $(\gamma_i)_{i \in I}$ of regularly semiopen L-fuzzy sets with $(\bigvee_{i \in I} \gamma_i)(x) \not\approx p$ for all $x \in X$ with $\lambda(x) \geq p'$, there is a finite subset $F$ of $I$ with $(\bigvee_{i \in F} \gamma_i)(x) \not\approx p$ for all $x \in X$ with $\lambda(x) \geq p'$.

**Theorem 2.8** Let $(X, \tau)$ be an L-fuzzy topological space and $\lambda \in L^X$. The L-fuzzy subset $\lambda$ is $S^*$-closed\(^{14}\) if and only if for every $p \in \Pr(L)$ and every collection $(\gamma_i)_{i \in I}$ of semiclopen L-fuzzy sets with $(\bigvee_{i \in I} \gamma_i)(x) \not\approx p$ for all $x \in X$ with $\lambda(x) \geq p'$, there is a finite subset $F$ of $I$ with $(\bigvee_{i \in F} \gamma_i)(x) \not\approx p$ for all $x \in X$ with $\lambda(x) \geq p'$.

**Definition 2.9** Let $(X, \tau)$ be an L-fuzzy topological space and $\lambda \in L^X$. The L-fuzzy subset $\lambda$ is said to be L-fuzzy isocompact\(^{5}\) if every countably compact and closed L-fuzzy subset of $\lambda$ is L-fuzzy compact.

If $\lambda$ is the whole space, then L-fuzzy topological space $(X, \tau)$ is isocompact.

**Definition 2.10** Let $(X, \tau)$ be an L-fuzzy topological space and $\lambda \in L^X$. The L-fuzzy subset $\lambda$ is said to be semi-isocompact\(^{8}\) if and only if every countably compact and closed L-fuzzy subset of $\lambda$ is semi compact. If $\lambda$ is the whole space, then the L-fuzzy topological space $(X, \tau)$ is also semi-is compact.

**Theorem 2.11**\(^{13}\) Let $(X, \tau)$ be an S-closed L-fuzzy topological space. Then each regularly open L-fuzzy subset in $(X, \tau)$ is S-closed.
**Theorem 2.12** Let $(X, \tau)$ and $(Y, \tau')$ be L-fuzzy topological spaces and let $f : (X, \tau) \to (Y, \tau')$ be an almost continuous, almost open mapping and let $\lambda$ be an S-closed L-fuzzy subset of $(X, \tau)$. Then $f(\lambda)$ is an S-closed L-fuzzy subset of $(Y, \tau')$.

**Proposition 2.13** Let $(X, \tau)$ and $(Y, \tau')$ be L-fuzzy topological spaces and let $f : (X, \tau) \to (Y, \tau')$ be a semi-irresolute mapping with $f^{-1}(y)$ is finite for every $y \in Y$. If $\lambda \in L^X$ is $S^*$-closed in $(X, \tau)$, then $f(\lambda)$ is $S^*$-closed in $(Y, \tau')$.

**Proposition 2.14** Let $(X, \tau)$ and $(Y, \tau')$ be L-fuzzy topological spaces and let $f : (X, \tau) \to (Y, \tau')$ be a semi-irresolute mapping with $f^{-1}(y)$ is finite for every $y \in Y$. If $\lambda \in L^X$ is $S^*$-closed in $(X, \tau)$, then $f(\lambda)$ is $S^*$-closed in $(Y, \tau')$.

**Proposition 2.15** Let $(X, \tau)$ and $(Y, \tau')$ be L-fuzzy topological spaces and let $f : (X, \tau) \to (Y, \tau')$ be a semi-irresolute mapping with $f^{-1}(y)$ is finite for every $y \in Y$. If $\lambda \in L^X$ is semicompact in $(X, \tau)$, then $f(\lambda)$ is $S^*$-closed in $(Y, \tau')$.

**Proposition 2.16** Let $(X, \tau)$ and $(Y, \tau')$ be L-fuzzy topological spaces and let $f : (X, \tau) \to (Y, \tau')$ be a semi-weekly continuous mapping with $f^{-1}(y)$ is finite for every $y \in Y$. If $\lambda \in L^X$ is semicompact in $(X, \tau)$, then $f(\lambda)$ is $S^*$-closed in $(Y, \tau')$.

**Theorem 2.17** If $\lambda$ is an L-fuzzy subset of $(X, \tau)$, $\mu$ is an L-fuzzy subset of $(Y, \tau')$ and $X$ is product related to $Y$, then

a) $\mathrm{Cl}(\lambda \times \mu) = \mathrm{Cl}\lambda \times \mathrm{Cl}\mu$ and

b) $\mathrm{Int}(\lambda \times \mu) = \mathrm{Int}\lambda \times \mathrm{Int}\mu$ hold.

**Definition 2.18** An L-fuzzy topological space $(X, \tau)$ is called fully stratified if for each $p \in L$, the L-fuzzy set which takes constant value $p$ at each point $x \in X$ belongs to $\tau$.

**Theorem 2.19** If $(X, \tau)$ be a compact L-fuzzy topological space and $(Y, \tau')$ be a fully stratified L-fuzzy topological space, then the projection mapping $P_Y : X \times Y \to Y$ is L-fuzzy perfect.

**ISO-S-CLOSEDNESS AND ISO-S*-CLOSEDNESS IN L-FUZZY TOPOLOGICAL SPACES**

**Definition 3.1** Let $(X, \tau)$ be an L-fuzzy topological space and $\lambda \in L^X$. The L-fuzzy subset $\lambda$ is said to be iso-S-closed if and only if every closed countably compact subset of $\lambda$ is S-closed.

If $\lambda$ is the whole space, then the L-fuzzy topological space $(X, \tau)$ is also iso-S-closed.
Theorem 3.2

Let \((X, \tau)\) be an L-fuzzy topological space and \(\lambda \in L^X\). Then the L-fuzzy subset \(\lambda\) is iso-S-closed if and only if every regular closed countably compact subset of \(\lambda\) is S-closed.

Proof: Since each regular closed set is closed then the result follows immediately from the definition 3.1.

Theorem 3.3

If an L-fuzzy topological space \((X, \tau)\) is the union of a countable collection of closed and iso-S-closed L-fuzzy subsets, then \((X, \tau)\) is L-fuzzy iso-S-closed.

Proof: Suppose \(X = \bigvee \mu_i\), where each \(\mu_i\) is closed and iso-S-closed L-fuzzy subset of \(X\) and let \(\beta\) be a closed and countably compact L-fuzzy subset of \(X\). Let \(p \in \text{pr}(L)\) and let \(\gamma_i\) be a family of semi-open L-fuzzy sets with \((\bigvee i \in J \gamma_i)(x) \not\geq p\) for all \(x \in X\) such that \(\beta(x) \geq p'\). For each \(i\), \(\beta \land \mu_i\) is a closed, countably compact L-fuzzy subset of \(\mu_i\). So it is S-closed L-fuzzy subset, since each \(\mu_i\) is L-fuzzy iso-S-closed. By S-closedness of \(\beta \land \mu_i\) there exist a finite subset \(F\) of \(J\) with \((\bigvee i \in F \text{cl} \gamma_i)(x) \not\geq p\) for all \(x \in X\) such that \((\beta \land \mu_i)(x) \geq p'\) i.e. \(\beta(x) \geq p'\). Hence \(\beta\) is a S-closed L-fuzzy subset, which implies that \(X\) is L-fuzzy iso-S-closed.

Theorem 3.4

Let \(f : (X, \tau) \to (Y, \tau')\) be an L-fuzzy perfect, almost continuous and almost open mapping from an iso-S-closed L-fuzzy topological space \((X, \tau)\) onto an L-fuzzy topological space \((Y, \tau')\). Then \((Y, \tau')\) is L-fuzzy iso-S-closed.

Proof: Let \(\beta\) be a regular closed and countably compact L-fuzzy subset of \((Y, \tau')\). Since \(f\) is L-fuzzy perfect map, then \(f^{-1}(\beta)\) is closed and countably compact L-fuzzy subset of \((X, \tau)\). By L-fuzzy iso-S-closedness of \((X, \tau)\), \(f^{-1}(\beta)\) is L-fuzzy S-closed. Since \(f\) is onto L-fuzzy almost continuous and almost open mapping then \(ff^{-1}(\beta) = \beta\) is S-closed [by 2.12] L-fuzzy subset in \((Y, \tau')\). Hence \((Y, \tau')\) is L-fuzzy iso-S-closed.

Theorem 3.5

If \((X, \tau)\) and \((Y, \tau')\) be two S-closed L-fuzzy topological spaces such that \(X\) is product related to \(Y\), then \(X \times Y\) is L-fuzzy S-closed.

Proof: Let \(\{\lambda_i \times \beta_i : i \in I\}\) be an L-fuzzy cover of \(X \times Y\) by semi-open L-fuzzy sets of \(X \times Y\), where \(\lambda_i\)'s and \(\beta_i\)'s are semi-open L-fuzzy sets in \(X\) and \(Y\) respectively. Then \(\{\lambda_i : i \in I\}\) and \(\{\beta_i : i \in I\}\) are L-fuzzy semi-open covers of \(X\) and \(Y\) respectively. As \((X, \tau)\) and \((Y, \tau')\) are S-closed L-fuzzy...
topological spaces then there exist finite subsets M and N of I such that, \((\forall i \in M (\text{cl} \lambda_i))\) \(x) \not\in p\) and
\((\forall i \in N (\text{cl} \beta_i))\) \(x) \not\in p\).

Now, \(\{\forall \text{cl}(\lambda_i \times \beta_i) : i \in M \lor N\}\) \((x) = [\forall \{\text{cl} \lambda_i : i \in M \lor N\}] \times [\forall \{\text{cl} \beta_i : i \in M \lor N\}]\) \((x) \not\in p\)

Hence the proof.

**Theorem 3.6**

Let \((X, \tau)\) be an \(S\)-closed L-fuzzy topological space and \((Y, \tau')\) be a fully stratified iso-\(S\)-closed L-fuzzy topological space such that \(X\) is product related to \(Y\). Then \(X \times Y\) is L-fuzzy iso-\(S\)-closed.

**Proof:** Let \((X, \tau)\) be an \(S\)-closed L-fuzzy topological space and \((Y, \tau')\) be a fully stratified iso-\(S\)-closed L-fuzzy topological space and consider the projection map \(P_Y : X \times Y \to Y\).

Let \(\beta\) be a countably compact, closed L-fuzzy subset of \(X \times Y\). \(P_Y(\beta)\) is countably compact and closed L-fuzzy subset as \(P_Y\) being L-fuzzy continuous. By L-fuzzy iso-\(S\)-closedness of \((Y, \tau')\), \(P_Y(\beta)\) is \(S\)-closed L-fuzzy subset of \((Y, \tau')\). Thus by 3.5, \(X \times P_Y(\beta)\) is L-fuzzy \(S\)-closed. So \(\beta\) is L-fuzzy countably compact, closed subset of \(X \times P_Y(\beta) \leq X \times Y\), and is \(S\)-closed L-fuzzy subset of \(X \times Y\). Hence \(X \times Y\) is L-fuzzy iso-\(S\)-closed.

**Definition 3.7** An L-fuzzy topological space \((X, \tau)\) is called hereditarily iso-\(S\)-closed if every subspace of it is iso-\(S\)-closed.

**Theorem 3.8**

Let \((X, \tau)\) be a fully stratified iso-\(S\)-closed L-fuzzy topological space and \((Y, \tau')\) be a hereditarily iso-\(S\)-closed L-fuzzy topological space such that \(X\) is product related to \(Y\). Then \(X \times Y\) is L-fuzzy iso-\(S\)-closed.

**Proof:** Let \((X, \tau)\) be a fully stratified iso-\(S\)-closed L-fuzzy topological space and \((Y, \tau')\) be a hereditarily L-fuzzy iso-\(S\)-closed space. Let us consider the projection map \(P_Y : X \times Y \to Y\).

Let \(\beta\) be a countably compact, closed L-fuzzy subset of \(X \times Y\). Then \(P_Y(\beta)\) is countably compact L-fuzzy subset of \((Y, \tau')\). Since \((Y, \tau')\) is hereditarily iso-\(S\)-closed L-fuzzy topological space then \(P_Y(\beta)\) is L-fuzzy \(S\)-closed. Thus from 3.6, \(X \times P_Y(\beta)\) is L-fuzzy iso-\(S\)-closed. Since \(\beta\) is a countably compact and closed subset of \(X \times P_Y(\beta) \leq X \times Y\), \(\beta\) is L-fuzzy \(S\)-closed subset of \(X \times Y\). Hence \(X \times Y\) is L-fuzzy iso-\(S\)-closed.

**Definition 3.9** Let \((X, \tau)\) be an L-fuzzy topological space and \(\lambda \in L^X\). The L-fuzzy subset \(\lambda\) is said to be iso- \(S^*\)-closed if and only if every closed countably compact subset of \(\lambda\) is \(S^*\)-closed.

If \(\lambda\) is the whole space, then the L-fuzzy topological space \((X, \tau)\) is also iso- \(S^*\)-closed.
Theorem 3.10

If an \( L \)-fuzzy topological space \((X, \tau)\) is the union of a countable collection of closed and iso- \( S^* \)-closed \( L \)-fuzzy subsets, then \((X, \tau)\) is \( L \)-fuzzy iso- \( S^* \)-closed.

**Proof:** Suppose \( X = \bigvee \mu_i \) where each \( \mu_i \) is closed and iso-\( S^* \)-closed \( L \)-fuzzy subset of \( X \) and let \( \beta \) be a closed and countably compact \( L \)-fuzzy subset of \( X \). Let \( p \in \text{pr} (L) \) and let \( \{\gamma_i\}_{i \in J} \) be a family of semi-open \( L \)-fuzzy sets with \( (\bigvee_{i \in J} (\gamma_i))(x) \nRightarrow p \) for all \( x \in X \) such that \( \beta(x) \geq p' \).

For each \( i \), \( \beta \land \mu_i \) is a closed, countably compact \( L \)-fuzzy subset of \( \mu_i \). So it is \( S^* \)-closed \( L \)-fuzzy subset, since each \( \mu_i \) is \( L \)-fuzzy iso- \( S^* \)-closed. By \( S^* \)-closedness of \( \beta \land \mu_i \), there exist a finite subset \( F \) of \( J \) with \( (\bigvee_{i \in F} (\text{cl} \ast \gamma_i))(x) \nRightarrow p \) for all \( x \in X \) such that \( \beta \land \mu_i (x) \geq p' \) i.e. \( \beta(x) \geq p' \).

Hence \( \beta \) is \( S^* \)-closed \( L \)-fuzzy subset, which implies that \( X \) is \( L \)-fuzzy iso- \( S^* \)-closed.

Theorem 3.11

Let \( f : (X, \tau) \to (Y, \tau') \) be an \( L \)-fuzzy perfect and semi-irresolute mapping with \( f^{-1}(y) \) is finite for every \( y \in Y \), from an iso- \( S^* \)-closed \( L \)-fuzzy topological space \((X, \tau)\) onto an \( L \)-fuzzy topological space \((Y, \tau')\). Then \((Y, \tau')\) is \( L \)-fuzzy iso- \( S^* \)-closed.

**Proof:** Let \( \beta \) be a closed and countably compact \( L \)-fuzzy subset of \((Y, \tau')\). Since \( f \) is \( L \)-fuzzy perfect map, then \( f^{-1}(\beta) \) is closed and countably compact \( L \)-fuzzy subset of \((X, \tau)\). By \( L \)-fuzzy iso- \( S^* \)-closedness of \((X, \tau)\), \( f^{-1}(\beta) \) is \( L \)-fuzzy \( S^* \)-closed. Since \( f \) is onto \( L \)-fuzzy semi-irresolute mapping with \( f^{-1}(y) \) is finite for every \( y \in Y \), then \( f f^{-1}(\beta) = \beta \) is \( S^* \)-closed \[by 2.13\] \( L \)-fuzzy subset in \((Y, \tau')\). Hence \((Y, \tau')\) is \( L \)-fuzzy iso- \( S^* \)-closed.

Theorem 3.12

Let \( f : (X, \tau) \to (Y, \tau') \) be an \( L \)-fuzzy perfect and irresolute mapping with \( f^{-1}(y) \) is finite for every \( y \in Y \), from an iso- \( S^* \)-closed \( L \)-fuzzy topological space \((X, \tau)\) onto an \( L \)-fuzzy topological space \((Y, \tau')\). Then \((Y, \tau')\) is \( L \)-fuzzy iso- \( S^* \)-closed.

**Proof:** Since every irresolute mapping is semi-irresolute \[14\], then the result is obvious from proposition \[2.14\].
Theorem 3.13

Let \( f : (X, \tau) \to (Y, \tau') \) be an L-fuzzy perfect and semi-irresolute mapping with \( f^{-1}(y) \) is finite for every \( y \in Y \), from an semi-iso-compact L-fuzzy topological space \( (X, \tau) \) onto an L-fuzzy topological space \( (Y, \tau') \). Then \( (Y, \tau') \) is L-fuzzy iso-S*+closed.

Proof: With the help of proposition 2.15, we can prove this theorem similarly as 3.11.

Theorem 3.14

If \( f : (X, \tau) \to (Y, \tau') \) be L-fuzzy perfect and semi weakly continuous mapping with \( f^{-1}(y) \) is finite for every \( y \in Y \), from an L-fuzzy semi-isocompact space \( (X, \tau) \) onto an L-fuzzy topological space \( (Y, \tau') \), then \( (Y, \tau') \) is L-fuzzy iso-S*+closed.

Proof: Let \( \beta \) be a closed and countably compact L-fuzzy subset of \( (Y, \tau') \). Since \( f \) is L-fuzzy perfect map, \( f^{-1}(\beta) \) is closed and countably compact L-fuzzy subset of \( (X, \tau) \). By L-fuzzy semi-compactness of \( (X, \tau) \), \( f^{-1}(\beta) \) is L-fuzzy semi-compact. By 2.16, \( ff^{-1}(\beta) = \beta \) is S*+closed L-fuzzy subset in \( (Y, \tau') \). Hence \( (Y, \tau') \) is L-fuzzy iso-S*+closed.

Theorem 3.15

If \( (X, \tau) \) and \( (Y, \tau') \) be two S*+closed L-fuzzy topological spaces such that \( X \) is product related to \( Y \), then \( X \times Y \) is L-fuzzy S*+closed.

Proof: Let \( \lambda \times \beta \) be an L-fuzzy cover of \( X \times Y \) by semi-open L-fuzzy sets of \( X \times Y \), where \( \lambda \)'s and \( \beta \)'s are semi-open L-fuzzy sets in \( X \) and \( Y \) respectively. Then \( \{ \lambda : i \in I \} \) and \( \{ \beta : i \in I \} \) are L-fuzzy semi-open covers of \( X \) and \( Y \) respectively. As \( (X, \tau) \) and \( (Y, \tau') \) are S*+closed L-fuzzy topological spaces then there exist finite subsets \( M \) and \( N \) of \( I \) such that, \( \bigvee_{i \in M} (\text{cl} \ast \lambda_i) (x) \neq p \) and \( \bigvee_{i \in N} (\text{cl} \ast \beta_i) (x) \neq p \). Now, \( \bigvee \text{cl} \ast (\lambda \times \beta) : i \in M \lor N \) (x) = \( \bigvee \text{cl} \ast \lambda_i : i \in M \lor N \} (x) \times \bigvee \text{cl} \ast \beta_i : i \in M \lor N \} (x) \neq p \). Hence the proof.

Theorem 3.16

Let \( (X, \tau) \) be an S*+closed L-fuzzy topological space and \( (Y, \tau') \) be a fully stratified iso-S*+closed L-fuzzy topological space such that \( X \) is product related to \( Y \). Then \( X \times Y \) is L-fuzzy iso-S*+closed.

Proof: Let \( (X, \tau) \) be an S*+closed L-fuzzy topological space and \( (Y, \tau') \) be a fully stratified iso-S*+closed L-fuzzy topological space and consider the projection map \( P_Y : X \times Y \to Y \).
Let $\beta$ be a countably compact, closed $L$-fuzzy subset of $X \times Y$. $P_Y(\beta)$ is countably compact and closed $L$-fuzzy subset as $P_Y$ being $L$-fuzzy continuous. By $L$-fuzzy iso-$S^*$-closedness of $(Y, \tau')$, $P_Y(\beta)$ is $S^*$-closed $L$-fuzzy subset of $(Y, \tau')$. Thus by 3.15, $X \times P_Y(\beta)$ is $L$-fuzzy $S^*$-closed. So $\beta$ is $L$-fuzzy countably compact, closed subset of $X \times Y$. Hence $X \times Y$ is $L$-fuzzy iso-$S^*$-closed.

**Definition 3.17** An $L$-fuzzy topological space $(X, \tau)$ is called hereditarily iso-$S^*$-closed if every subspace of it is iso-$S^*$-closed.

**Theorem 3.18**

Let $(X, \tau)$ be a fully stratified iso-$S^*$-closed $L$-fuzzy topological space and $(Y, \tau')$ be a hereditarily iso-$S^*$-closed $L$-fuzzy topological space such that $X$ is product related to $Y$. Then $X \times Y$ is $L$-fuzzy iso-$S^*$-closed.

**Proof:** Let $(X, \tau)$ be a fully stratified iso-$S^*$-closed $L$-fuzzy topological space and $(Y, \tau')$ be a hereditarily $L$-fuzzy iso-$S^*$-closed space. Let us consider the projection map $P_Y : X \times Y \rightarrow Y$.

Let $\beta$ be a countably compact, closed $L$-fuzzy subset of $X \times Y$. Then $P_Y(\beta)$ is countably compact $L$-fuzzy subset of $(Y, \tau')$. Since $(Y, \tau')$ is hereditarily iso-$S^*$-closed $L$-fuzzy topological space then $P_Y(\beta)$ is $L$-fuzzy $S^*$-closed. Thus from 3.16, $X \times P_Y(\beta)$ is $L$-fuzzy iso-$S^*$-closed. Since $\beta$ is a countably compact and closed subset of $X \times P_Y(\beta) \leq X \times Y$, $\beta$ is $L$-fuzzy $S^*$-closed subset of $X \times Y$. Hence $X \times Y$ is $L$-fuzzy iso-$S^*$-closed.

**Definition 3.19** An $L$-fuzzy topological space $(X, \tau)$ is said to be extremally disconnected if and only if $\text{cl}(\lambda) \in \tau$ for every $\lambda \in \tau$.

Kudri established a relation among semi-compact space, s-closed and $S^*$-closedness in $L$-fuzzy topological spaces.

**Proposition 3.20** Semi-compactness $\Rightarrow S^*$-closedness $\Rightarrow S$-closedness.

**Theorem 3.21**

Let $(X, \tau)$ be an $L$-fuzzy topological space. Then the following relations hold. $(X, \tau)$ is Semi-iso-compact $\Rightarrow (X, \tau)$ is iso-$S^*$-closed $\Rightarrow (X, \tau)$ is iso-$S$-closed.

**Proof:** The proof immediately follows from the proposition [3.20].
Theorem 3.22

Let \((X, \tau)\) be an extremally disconnected L-fuzzy topological space and \(\lambda \in L^X\). Then the following are equivalent:

i) \(\lambda\) is almost compact \(^{12}\).

ii) \(\lambda\) is nearly compact \(^{14}\).

iii) \(\lambda\) is S-closed.

iv) \(\lambda\) is \(S^*\)-closed.

v) \(\lambda\) is \(SS\)-closed \(^4\).

Theorem 3.23

Let \((X, \tau)\) be an extremally disconnected L-fuzzy topological space and \(\lambda \in L^X\). Then the following are equivalent:

i) \(\lambda\) is weakly iso-compact \(^6\).

ii) \(\lambda\) is nearly iso-compact \(^7\).

iii) \(\lambda\) is iso-S-closed.

iv) \(\lambda\) is iso-\(S^*\)-closed.

v) \(\lambda\) is iso-SS-closed \(^9\).

Proof: First of all, we show that (i) \(\Rightarrow\) (ii). Suppose, \((X, \tau)\) is weakly iso-compact and extremally disconnected L-fuzzy topological space. Let \(\lambda\) be a regular closed, countably almost compact L-fuzzy subset of \((X, \tau)\). Since countably almost compact extremally disconnected L-fuzzy topological space is countably nearly compact then \(\lambda\) is countably nearly compact. As \((X, \tau)\) is L-fuzzy weakly isocompact, \(\lambda\) is almost compact and hence nearly compact ( almost compact extremally disconnected L-fuzzy topological space is nearly compact ). Hence \((X, \tau)\) is nearly isocompact L-fuzzy topological space.

(ii) \(\Rightarrow\) (iii), (iii) \(\Rightarrow\) (iv), (iv) \(\Rightarrow\) (v) and (v) \(\Rightarrow\) (i) can be proved similarly.

Corollary 3.24

Let \((X, \tau)\) be an extremally disconnected L-fuzzy topological space. If \(\lambda \in L^X\) is compact then \(\lambda\) is S-closed \((S^*\)-closed\).

Corollary 3.25

Let \((X, \tau)\) be an extremally disconnected L-fuzzy topological space. If \(\lambda \in L^X\) is iso-compact then \(\lambda\) is iso-S-closed \((iso-S^*\)-closed\).
Proof: Let $\lambda$ be a closed countably compact L-fuzzy subset in $(X, \tau)$. Since it is isocompact then $\lambda$ is compact. From 3.24, an extremally disconnected compact space is $S$-closed ($S^*$-closed) and so $\lambda$ is $S$-closed ($S^*$-closed) and consequently it is isocompact (iso-$S^*$-closed).

4. CL- ISO-S-CLOSEDNESS AND CL- ISO-$S^*$-CLOSEDNESS IN L-FUZZY TOPOLOGICAL SPACES

M. Sakai\(^{16}\) introduced and studied CL-isocompactness (spaces in which closure of each countably compact subspace is compact) in classical topology. In this section considering the S-closedness and $S^*$-closedness in L-fuzzy topological spaces, a generalized stronger form of iso-S-closedness and iso-$S^*$-closedness are introduced and these new class of L-fuzzy topological spaces are called L-fuzzy CL-iso-S-closed and CL-iso$S^*$-closed spaces. Some properties of these spaces are studied here.

**Definition 4.1** Let $(X, \tau)$ be an L-fuzzy topological space. The L-fuzzy set $\lambda \in L^X$ is said to be L-fuzzy CL-iso-S-closed if the closure of each L-fuzzy countably compact subspace of $\lambda$ is L-fuzzy S-closed. If $\lambda$ is the whole space, then we say that the L-fuzzy topological space $(X, \tau)$ is L-fuzzy CL-iso-S-closed. Obviously every L-fuzzy CL-iso-S-closed spaces are L-fuzzy iso-S-closed.

**Theorem 4.2**

Let $f : (X, \tau) \rightarrow (Y, \tau')$ be an L-fuzzy perfect, almost continuous and almost open mapping with $f^{-1}(y)$ is finite for every $y \in Y$, from a CL-iso-S-closed space $(X, \tau)$ onto an L-fuzzy topological space $(Y, \tau')$. Then $(Y, \tau')$ is CL-iso-S-closed L-fuzzy topological space.

**Proof:** Let $\beta$ be an L-fuzzy countably compact subset of $(Y, \tau')$. Since $f$ is L-fuzzy perfect, $f^{-1}(\beta)$ is L-fuzzy countablycompact subset of $(X, \tau)$. As $(X, \tau)$ is L-fuzzy CL-iso-S-closed then $cl(f^{-1}(\beta))$ is L-fuzzy S-closed. Since $f$ is L-fuzzy closed, continuous and onto then $f(cl(f^{-1}(\beta))) = cl(f(f^{-1}(\beta))) = cl(\beta)$, which implies $cl(\beta)$ is L-fuzzy S-closed. Hence $(Y, \tau')$ is L-fuzzy CL-iso-S-closed.

**Theorem 4.3**

Let $(X, \tau)$ be a fully stratified L-fuzzy iso-S-closed space and $(Y, \tau')$ be an L-fuzzy CL-iso-S-closed space such that X is product related to Y. Then $X \times Y$ is L-fuzzy CL-iso-S-closed.

**Proof:** Let $(X, \tau)$ be a fully stratified iso-S-closed L-fuzzy topological space and $(Y, \tau')$ be an L-fuzzy CL-iso-S-closed space. Let $P_X : X \times Y \rightarrow X$ and $P_Y : X \times Y \rightarrow Y$ be the projection maps.

Let $\beta$ be an L-fuzzy countably compact subset of $X \times Y$. Then $P_Y(\beta)$ is L-fuzzy countably compact. By L-fuzzy CL-iso-S-closedness of $(Y, \tau')$, $cl(P_Y(\beta))$ is L-fuzzy S-closed in $(Y, \tau')$. But $P_X(\beta)$ is L-fuzzy countably compact subset of $X$. Since $X$ is fully stratified, then by 2.19, $P_X$
is L-fuzzy perfect and so is L-fuzzy closed. So $P_X(\beta)$ is L-fuzzy S-closed in $X$. Thus $\text{cl}(\beta)$ is contained in the L-fuzzy S-closed space $P_X(\beta) \times \text{cl}(P_Y(\beta))$ (by 3.5). Hence $\text{cl}(\beta)$ is L-fuzzy S-closed, which follows that $X \times Y$ is L-fuzzy CL-iso-S-closed.

**Definition 4.4** An L-fuzzy topological space $(X, \tau)$ is called hereditarily CL-iso-S-closed if every subspace of it is CL-iso-S-closed.

**Theorem 4.5**

If an L-fuzzy topological space $(X, \tau)$ is hereditarily CL-iso-S-closed then $(X, \tau)$ is hereditarily iso-S-closed.

**Proof:** Since every CL-iso-S-closed L-fuzzy topological space is L-fuzzy iso-S-closed, the result follows immediately.

**Definition 4.6** Let $(X, \tau)$ be an L-fuzzy topological space. The L-fuzzy set $\lambda \in L^X$ is said to be L-fuzzy CL-iso- $S^*$-closed if the closure of each L-fuzzy countably compact subspace of $\lambda$ is L-fuzzy $S^*$-closed. If $\lambda$ is the whole space, then we say that the L-fuzzy topological space $(X, \tau)$ is L-fuzzy CL-iso- $S^*$-closed.

Obviously every L-fuzzy CL-iso- $S^*$-closed spaces are L-fuzzy iso- $S^*$-closed.

**Theorem 4.7**

Let $f : (X, \tau) \rightarrow (Y, \tau')$ be an L-fuzzy perfect and semi-irresolute mapping with $f^{-1}(y)$ is finite for every $y \in Y$, from a CL-iso- $S^*$-closed space $(X, \tau)$ onto an L-fuzzy topological space $(Y, \tau')$. Then $(Y, \tau')$ is CL- iso- $S^*$-closed L-fuzzy topological space.

**Proof:** Let $\beta$ be an L-fuzzy countably compact subset of $(Y, \tau')$. Since $f$ is L-fuzzy perfect, $f^{-1}(\beta)$ is L-fuzzy countablycompact subset of $(X, \tau)$. As $(X, \tau)$ is L-fuzzy CL-iso- $S^*$-closed then $\text{cl}(f^{-1}(\beta))$ is L-fuzzy $S^*$-closed. Since $f$ is L-fuzzy closed, continuous and onto then $\text{cl}(f^{-1}(\beta)) = \text{cl}(\beta)$, which implies $\text{cl}(\beta)$ is L-fuzzy $S^*$-closed. Hence $(Y, \tau')$ is L-fuzzy CL-iso- $S^*$-closed.

**Theorem 4.8**

Let $(X, \tau)$ be a fully stratified L-fuzzy iso-$S^*$-closed space and $(Y, \tau')$ be an L-fuzzy CL- iso-$S^*$-closed space such that $X$ is product related to $Y$. Then $X \times Y$ is L-fuzzy CL- iso-$S^*$-closed.

**Proof:** Let $(X, \tau)$ be a fully stratified iso-$S^*$-closed L-fuzzy topological space and $(Y, \tau')$ be an L-fuzzy CL- iso-$S^*$-closed space. Let $P_X : X \times Y \rightarrow X$ and $P_Y : X \times Y \rightarrow Y$ be the projection maps.
Let $\beta$ be an L-fuzzy countably compact subset of $X \times Y$. Then $P_Y(\beta)$ is L-fuzzy countably compact. By L-fuzzy CL- iso- $S^*$-closedness of $(Y, \tau')$, $cl(P_Y(\beta))$ is L-fuzzy $S^*$-closed in $(Y, \tau')$. But $P_X(\beta)$ is L-fuzzy countably compact subset of $X$. Since $X$ is fully stratified, then by 2.19, $P_X$ is L-fuzzy perfect and so is L-fuzzy closed. So $P_X(\beta)$ is L-fuzzy $S^*$-closed in $X$. Thus $cl(\beta)$ is contained in the L-fuzzy $S^*$-closed space $P_X(\beta) \times cl(P_Y(\beta))$ (by 3.15). Hence $cl(\beta)$ is L-fuzzy $S^*$-closed, which follows that $X \times Y$ is L-fuzzy CL-iso- $S^*$-closed.

**Definition 4.9**

An L-fuzzy topological space $(X, \tau)$ is called hereditarily CL-iso- $S^*$-closed if every subspace of it is CL-iso- $S^*$-closed.

**Theorem 4.10**

If an L-fuzzy topological space $(X, \tau)$ is hereditarily CL-iso-$S^*$-closed then $(X, \tau)$ is hereditarily iso-$S^*$-closed.

**Proof:** Since every CL-iso- $S^*$-closed L-fuzzy topological space is L-fuzzy iso- $S^*$-closed, the result follows immediately.

**CONCLUSION**

There are so many compactness in literature. At present, many authors are working on various forms of compactness in fuzzy topology as well as L-fuzzy topological spaces. Our intention is to generalize these compactness in L-fuzzy topological spaces and to establish relations with other spaces. As a result we have defined concepts iso-$S$-closedness and iso- $S^*$-closedness and their stronger forms CL-iso-$S$-closedness and CL-iso- $S^*$-closedness and got some important relations with other spaces. We feel that with the help of these spaces some more new spaces will be developed in future.

**REFERENCES:**


