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Path Induced Geodesic Graphs

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ABSTRACT

Let G be a connected graph with at least two vertices. A connected geodetic set $S \subseteq V(G)$ is said to be a *path induced geodetic (pig) set of* Gif $< S >$ contais a path P, where $V(P) = S$. The minimum cardinality of a path induced geodetic set of G is called a *path induced geodetic number* of$G$ and is denoted by $pign(G)$. Some properties satisfied by this concept are studied. It is prove that $pign(G) \geq 1 + d$. It is shown that for any positive integers $2 \leq d < p$, there exists a path induce geodesic graph G such that $pign(G) = 1 + d$, where d is the diameter of G and p is the order of G.In this paper we investigate how the path induced geodetic number is affected by adding a pendant edge to G. It is proved that if $G'$is a graph obtained from G by adding a pendant edge, then $pign(G') \geq pign(G) + 1$.

KEYWORDS: geodesic, geodetic number, connected geodetic number, path induced geodeticnumber, path induced geodesicgraphs.

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1. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer to Harary [1]. The distance $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest $u\text{–}v$ path in G. An $u\text{–}v$ path of length $d(u, v)$ is called an $u\text{–}v$ geodesic. A vertex x is said to lie on a $u - v$ geodesic P if x is a vertex of P including the vertices u and v. The *eccentricity* $e(v)$ of a vertex v in G is the maximum distance from v and a vertex of G. The minimum eccentricity among the vertices of G is the *radius*, *rad* Gorr$(G)$and the maximum eccentricity is its *diameter*, *diam* G of G.Twovertices u and v of G are *antipodal* if $d(u, v) = \ diam\, G$ or $d(G)$. A vertex v is said to be an *extreme vertex* if the subgraph induced by its neighbours is complete. A $geodetic\ set$ of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The $geodetic\ number$ $g(G)$ of G is the minimum order of its geodetic sets and geodetic set of order $g(G)$ is called $g - set$ or $geodetic\ basis$. The geodetic number of a graph was introduced and studied in 1,2, 3, 4. A *connected geodetic set* of a graph G is a geodetic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected geodetic set of G is the *connected geodetic number* of G and is denoted by $g_c(G)$. A connected geodetic set of cardinality $g_c(G)$ is called a g_c*-set* of G or a *connected geodetic basis* of G. The connected geodetic number of a graph was introduced and studied in [5, 7, 8]. A connected geodetic set $S \subseteq V(G)$ is said to be a *path induced geodetic(pig) set of* G if $< S >$ contains a path P with $V(P) = S$. The minimum cardinality of a path induced geodetic set of G is called a *path induced geodetic number* of G and is denoted by $pign(G)$. The path induced geodetic number of a graph was introduced in and studied[6]. The concept on path-induced geodetic numbers of graphs can be applied in travel time saving, facility location, goods distribution, and other things in which this concept will be of great help. First, we have to remark that not all connected graphs have path-induced geodetic set.Note that the path induced geodetic set does not exists for all connected graphs. For example a tree with more than two vertices do not have a the path induced geodetic set. This motivates us to define a path induced geodesic graphs. The following theorems are used in sequel.

Theorem 1.1[1]. For a connected graph G, $g(G) = 2$ if and only if there exist peripheral vertices u and v such that every vertex of G is on a diametral path joining u and v.

Theorem1.2[5]. Eachextreme vertex of a graph connected G belongs to every path induced geodetic set of G. In particular, each end-vertex of G belongs to every path induced geodetic set of G.

Theorem1.3[5]. Every cut vertex of a connected graph G belongs to every path induced geodetic set of G.

Theorem 1.4.[6] For a connected graph $G, g_c(G) \geq 1 + d,$ where d is the diameter of G.

2.PATH INDUCED GEODESIC GRAPHS

Definition 2.1. A connected graph G is said to be a *path induced geodesic graph* if G has a path induced geodetic set.

Example 2.2. The complete graph, cycle, path graph, complete bipartite graph $K_{m,n}$ $(4 \leq m \leq n)$ are some examples of path induced geodesic graphs. A connected graph with more than two end edges is not a path induced geodesic graph.

Theorem 2.3. Let G be a path induced geodesic graph. Then$2 \leq g_c(G) \leq pign(G) \leq p$.

Proof. A connected geodetic set needs at least two vertices. Therefore $g_c(G) \geq 2$. Since every path induced geodetic set of Gisaconnectedgeodetic set of G, we have $g_c(G) \leq pig(G)$. Since $V(G)$ has a spanning path, $V(G)$ is a path induced geodetic set and so $pign(G) \leq p$. Thus $2 \leq g_c(G) \leq pign(G) \leq p$. ■

Example 2.4. The bounds in Theorem 2.3 is sharp. For $G = K_2$, $g_c(G) = 2$. For the graph G given in Figure 2.1, $S_1 = \{v_2, v_3, v_4, v_5\}$, $S_2 = \{v_1, v_2, v_3, v_5\}$and $S_3 = \{v_1, v_2, v_3, v_4\}$ are the only three g_c-sets of G so that $g_c(G) = 4$. Also S_1 and S_2 the only two pig-sets of G so that $pign(G) = 4$. Thus $g_c(G) = pign(G)$. For the path $G = P_p$, $pign(G) = p$. Also the bounds in Theorem 2.3 is strict. For the wheel $G = K_1 + C_{6,}$ with x as vertex set of K_1 and the vertex set of $C_6: v_1, v_2, v_3, v_4, v_5, v_6, S = \{x, v_1, v_3, v_5\}$ is a g_c-sets of G so that $g_c(G) = 4$ and $S = \{v_1, v_2, v_3, v_4, v_5\}$ is a pig-set of G so that $pig(G) = 5$. Thus $2 < g_c(G) < pign(G) < p$.

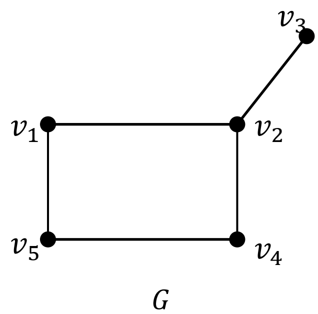


Figure 2.1

Theorem 2.5. For the wheel $G = K_1 + C_{p-1,}(p \geq 4)$, $pign(G) = p - 2$.

Proof : Let C_{p-1} be a $v_1, v_2, \ldots, v_{p-1}$ and x be the vertex of K_1. Then x is a vertex of degree $p - 1$. Let $S = \{v_1, v_2, \ldots, v_{p-2}\}$. Then S is a connected geodetic set of G. Since $< S >$ contains a path

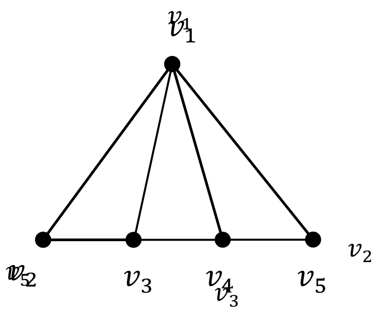
$P: v_1, v_2, \ldots, v_{p-2}, S$ is a path induced geodetic set of G so that $pign(G) \leq p-2$. We show that $pig(G) = p-2$. Suppose that $pign(G) \leq p-3$. Then there exists a path induced geodetic set S' with $|S'| \leq p-3$. If $x \in S'$, then there is no path in $P' < S' >$ with $V(P') = S'$. If $x \notin S'$, then $< S' >$ is not connected, which is a contradiction to S' a path induced geodetic set of G. Therefore $pign(G) = p-2$. ■

Theorem. 2.6. Let G be a path induced geodesic graph with exactly one vertex of degree $p-1$ which is not a cut vertex of G. Then $pign(G) \leq p-1$.

Proof : Let x be a vertex of degree $p-1$. Then $S = V - \{x\}$.Since S is not a cut vertex of G, $< S >$ is connected and $< S >$ contains a path P with $V(P) = S$. Hence S is a path induced geodetic set of G so that $pign(G) \leq p-1$. ■

Remark, 2.7. The bound in Theorem 2.6 is sharp. For the graph G given in Figure 2.2, $S = \{v_2, v_3, v_4, v_5\}$ is a pig-set of G so that $pign(G) = 4$. Since $p = 5$, We have $pign(G) = p-1$. Also the bound in Theorem 2.4 is strict. For the wheel $G = K_1 + C_{p-1,}(p \geq 4)$, $pign(G) = p-2$.

v_1



G

Figure 2.2

Theorem 2.8. Let G be a path induced geodesic graph with a vertex x of degree $p-1$, which is a cut vertex of G. Then every path induced geodetic set of G contains every neighbour of x.

Proof : If each neighbour of x is an extreme vertex of G, then the result follows from Theorem 1.2. If not there exists $u, v \in V$ such that u and v are not adjacent. Let w be a vertex of G in $u - v$ geodesic u, w, v with $w \neq x$. Let S be a path induced geodetic set of G. Then by Theorem 1.3, $x \in S$. Suppose that $w \notin S$. Then $< S >$ is either a star or disconnected. Then there is is no path P such that $V(P) = S$. Therefore S is not a path induced geodetic set of G, which is a contradiction. Therefore every path induced geodetic set of G contains every neighbour of x. ■

Theorem 2.9. Let G be a path induced geodesic graphwith a vertex of degree $p-1$, which is a cut vertex of G. Then $pign(G) = p$ if and only if $G - x$ contains exactly two components.

Proof. Let x bea cut vertex of G which is of degree $p-1$.Let $pign(G) = p$. Then $S = V(G)$ is the unique pig-set of G. Let P be a path in $< S >$ with $V(G) = S$.Suppose that $G - x$ contains more than two components. Then$x$ occur more than once in the path P so that P is not a path of G, which is a contradiction. Therefore $G - x$ contains exactly two components. Conversly, let $G - x$ contains exactly two components. We prove that $pign(G) = p$. Let S be a pig-set of G. By Theorems 1.3 and 2.8 , $S = V(G)$.Since$G - x$ contains exactly two components, there is a path P in $< S >$ with $V(P) = S$. Hence $S = V(G)$ is the unique pig-set of G so that $pign(G) = p$. ■

Theorem2.10. Let G be a path induced geodesic graph. Then $pign(G) \geq 1 + d$,where d is the diameter of G.

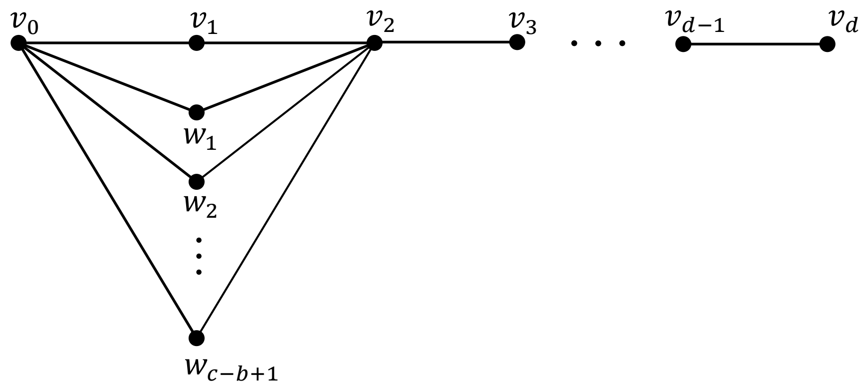
Proof. This follows from Theorems 1.4 and 2.3. ■

Theorem2.11. Let G be a path induced geodesic graphsuch that $g(G) = 2$, then $pign(G) = 1 + d$.

Proof. Let $g(G) = 2$. Then byTheorem 1.1, there exist peripheral vertices u and v such that every edge of G lies on a diametral path joining u and v. Let $P : u = u_0, u_1, u_2, \dots, u_n = v$ be a diametral path of G. Let $S = \{u_0, u_1, u_2, \dots, u_n\}$. Then it is clear that S is a path induced geodetic set of G so that $pign(G) \leq |S| = 1 + diam(G)$. Now the theorem follows from Theorem 2.10. ■

Theorem 2.12. For any positive integers $2 \leq d < p$, there exists a apath induced geodesic graph$G$ such that $pig(G) = 1 + d$,where d is the diameter of G and$p$is the order of$G$.

Proof. let $P_d : v_0, v_1, v_2, \dots, v_d$ be a path of length d.Let G be a graph obtained from P_d by adding $p - d$ new vertices $u_1, u_2, u_3, \dots, u_{p-d}$ to P_d and join each $u_i (1 \leq i \leq p - d)$ to v_0 and v_3 there by producing the graph G of Figure 2.3 such that the diameter of G is d andthe order of$G$is$p$. Let $S = \{v_2, v_3, \dots, v_d\}$ be the set of all end vertices and cut vertices of G. By Theorems 1.2and 1.3, every path induced geodetic set of G contains S. It is clear that S is not a path induced geodetic set G.It is easily verified that$S \cup \{x\}$ is not a path induced geodetic set G and so $pign(G) \geq 1 + d$. Let $S' = S \cup \{x\}$.Then $< S' >$ is connected and S' contains a path P_d with $V(P_d) = S'$. Since each vertex of G lies on a $x - y$ geodesic with$x, y \in S'$, S' is a path induced geodetic set G so that $pign(G) = 1 + d$. ■



G

Figure 2. 3

Theorem 2.13..Let G be a path induced geodesic graph with at most one cut end edge. Let G' be a graph obtained from G by adding an end edge xy at a vertex x which is not a cut vertex of G. Then $pign(G') \geq pign(G) + 1$.

Proof. Consider a *pig*-set S of G. Let P be a path on G such that $V(P) = S$. Let $u \in S$ and let $P': y, x, x_1, x_2, \dots x_n = u$, be a path on G. Then $S' = S \cup \{y_1 x, x_1, x_2, \dots x_n, y\}$ is a connected geodetic set of G. Let $P'' = P \cup P'$. Since $V(P'') = S', S'$is a path induced geodesic set of G' and so $pign(G') \geq pign(G) + 1$. ∎

Theorem 2.14. Let G be a path induced geodesic graph with at most one end edge. Let G' be a graph obtained from G by adding an end edge xy at a vertex x, which is not a cut vertex of G. Then $pign(G') = pign(G) + 1$ if and only if x is a vertex of some *pig*-set of G.

Proof. First, assume that there is a *pig*-set S of G such that $x \in S$. Then there a path P in $< S >$such that $V(P) = S$. Since y is an end edge of G, x is a cut vertex of G, by Theorems 1.2 and 1.3, x and y belong every path induced geodetic set of G'. Let $S' = S \cup \{y\}$. We show that S' is a pig-set of G'. Let P' be a path in G' such that $V(P') = V(P) \cup \{y\}$. Therefore $V(P') = S'$. We show that S' is a connected geodetic set of G. Let u be a vertex of G'. If $u = x$ or y, then it is clear that u lies on every $x - y$ geodesic in G. Therefore $u \neq x$ and $u \neq y$. Then it follows that u is a vertex of G. Since S is a pig-set of G, u lies on a $w - z$ geodesic in G with $w, z \in S$. If both $w, z \in S \cup \{y\}$, then u also lies on a $w - z$ geodesic in G' with $w, z \in S'$. If u lies on a $w - x$ geodesic in G with $w \in S \cup \{y\}$, then u also lies on $x - w$ geodesic in G'. Thus S' is a geodetic set of G'. Since $< S' >$ is connected and $V(P') = V(P) \cup \{y\} = S'$, S' is a path induced geodetic set of G' and so that $pign(G') \leq |S'| = |S \cup \{y\}| = pign(G) + 1$. Then from

Theorem 2.13 we have $pign(G') = pign(G) + 1$. Conversely, suppose that $pign(G') = pign(G) + 1$. Suppose that x does not belong to any pig-set of G. Let S' be an pig-set of G'. Since y is an end vertex of G' and x is a cut vertex of G', by Theorems 1.2 and 1.3, $x, y \in S'$. Let $S = S' - \{y\}$ Then $S \subseteq V(G)$ and $pign(G') = |S'| = |S| + 1 = pign(G) + 1$. Let u be any vertex of G. Then u is also a vertex of G' and so u lies on a geodesic P in G' joining a pair of vertices $w, z \in S'$. If $w \neq u$ and $z \neq u$, then $w \in S$ and $z \in S$ so that $u$lies on a geodesicjoining a pair of vertices in S. Otherwise, let $w \neq u$ and $z = u$. Then it follows that u lies on a geodesic in G joining w and x in S. Thus, S is a geodetic set of G and since $|S| = pign(G)$, it follows that S is an pig-set of G. Since $x \in S$, this is contradiction to our assumption. This completes the proof. ■

Corollary 2.15. Let G' be a graph obtained from $G = K_p (p \geq 2)$ or $G = C_p (p \geq 4)$ or $G = K_{m,n} (4 \leq m \leq n)$ by adding an end edge xy at a vertex x. Then $pign(G') = pign(G) + 1$.

Proof. This follows from Theorem 2.14. ■

Corollary 2.16. Let G' be a graph obtained from $G = P_p (p \geq 2)$ by adding an end edge xy at an end vertex x. Then $pign(G') = pign(G) + 1$.

Proof. Since G' is the path $P_{p+1} (p \geq 2)$, we have $pign(G') = p + 1 = pign(G) + 1$. ■

Theorem 2.17. Let G be a graph obtained from the cycle C_{p-2} $(p \geq 5)$ by attaching two end edges to two adjacent vertices of C_{p-2} $(p \geq 5)$. Then $pign(G) = p$.

Proof. Let C_{p-2} be $v_1, v_2, \ldots, v_{p-2}$.Without loss of generality, let x and y be attached with v_1 and v_2. Then $P_1 : x, v_1, v_2, y$ and $P_2 : x, v_1, v_{p-2}, \ldots, v_2, y$ are the only paths in G. Let $= \{x, v_1, \ldots, v_{p-2}, v_2, y\}$. Then$S$ is a connected geodetic set of G. Since $V(P_2) = S, S$ is a path induced geodetic set of G so that $pign(G) \leq p$.We prove that $pign(G) = p$. If $pig(G) < p$, then there exists a path induced geodetic set S' with $|S'| < p$. Then it is easily verified that there is no path P' with $V(P') \neq S'$. Hence $pign(G) = p$. ■

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