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Wiener and Hyper-Wiener polynomials of Unitary Cayley Graphs

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ABSTRACT:

The two generating functions, namely, Wiener and Hyper-Wiener polynomials are the q-analogues of the topological indices - Wiener and Hyper-Wiener indices respectively. Both polynomials have found substantial applications in chemical graph theory. However, these applications are by no means restricted to molecular graph, but we can also determine a remarkable variety of novel mathematical results. Motivated by this, we computed Wiener and Hyper-Wiener polynomials of Unitary Cayley graphs in this paper.

KEYWORDS: Wiener index, Wiener polynomial, Hyper-Wiener index, Hyper-Wiener polynomial, Unitary Cayley graphs.

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INTRODUCTION:

Throughout this paper, we consider simple connected graph \( G = (V, E) \) with \( n \) vertices and \( m \) edges. We denote the distance between the vertices \( u \) and \( v \) with \( d(u, v) \).

The Wiener polynomial of \( G \), \( W(G; q) \), is the polynomial whose first derivative at \( q = 1 \) gives the Wiener index. i.e., \( W(G) = W'(G;1) \).

Analogously, the Hyper-Wiener polynomial of \( G \), \( WW(G; q) \), is the polynomial whose first derivative at \( q = 1 \) gives the Hyper-Wiener index. i.e., \( WW(G) = WW'(G;1) \).

For more detailed study of these polynomials and their respective indices, refer \(^{2-9}\).

In this paper, we urge to find out the Wiener and Hyper-wiener polynomials of Unitary Cayley graphs. Given a positive integer \( n > 1 \), the Unitary Cayley graph, denoted by \( X_n \), can be defined as \( X_n = Cay(Z_n,U_n) \), where \( Z_n \) is the additive group of ring of integers modulo \( n \) and \( U_n \) is the multiplicative group of its units. Therefore, its vertex set is \( Z_n \) and edge set is \( \{(u,v);gcd(u-v,n)=1\} \), for \( u,v \in Z_n \). These graphs have got the property that they have integral spectrum and thus play a vital role in modelling quantum spin network supporting the perfect state transfer. Let \( \phi(n) \) denotes the Euler function. View \(^{1,10-13,15}\) for the comprehensive study of graphs and Unitary Cayley Graphs.

Let us see the following lemma which we use in the theorems:

**LEMMA 1.1:** \(^{[11]}\) Denote \( F_n(s) = F_n(a-b) \), the number of common neighbours of vertices \( a \neq b \) in the Unitary Cayley graph \( X_n \) for integers \( a, b, n \geq 2 \) and prime \( p \). Then \( F_n(s) \) is given by

\[
F_n(s) = n \prod_{p \mid n} \left( 1 - \frac{\varepsilon(p)}{p} \right), \quad \text{where } \varepsilon(p) = \begin{cases} 1, & \text{if } p \text{ divides } s \\ 2, & \text{if } p \text{ does not divide } s \end{cases}
\]

**WIENER POLYNOMIAL OF UNITARY CAYLEY GRAPHS:**

**THEOREM 2.1:** If \( X_n \) is the Unitary Cayley graph, then the Wiener polynomial of \( X_n \) is given by
\[
W(X_n; q) = \begin{cases} 
\frac{n(n-1)}{2} q, & \text{if } n \text{ is prime} \\
\frac{n\phi(n)}{2} q + \frac{n(n-2)}{4} q^2, & \text{if } n = 2^\alpha, \alpha > 1 \\
\frac{n\phi(n)}{2} q + \frac{n(n-2)}{4} q^2 + \frac{n(n-2\phi(n))}{4} q^3, & \text{if } n \text{ is even and has an odd prime divisor} \\
\frac{n\phi(n)}{2} q + \frac{n(n-\phi(n)-1)}{2} q^2, & \text{if } n \text{ is odd but not prime.}
\end{cases}
\]

**PROOF:** For \( n \) is prime, \( X_n \) is complete. So \( d(u, v) = 1, \forall \, u, v \in X_n \). Therefore, by definition of Wiener polynomial, we obtain \( W(X_n; q) = \sum_{\{u, v\}} q d(u, v) = \frac{n(n-1)}{2} q. \)

When \( n = 2^\alpha, \alpha > 1 \), \( X_n \) is complete bipartite with vertex partition \( V(X_n) = \{0, 2, \ldots, (n-2)\} \cup \{1, 3, \ldots, (n-1)\} \). Then it is clear that \( d(u, v) = 1 \) or 2. As a result, we get a 2-degree polynomial such that \( W(X_n; q) = n^2 q + n(n-1)q^2. \)

Now we take the case of \( n \) as even and has an odd prime divisor \( p \), where \( n \neq 2^\alpha, \alpha > 1 \). This shows that \( X_n \) is bipartite with vertex set \( V \) as the union of \( V_1 = \{0, 2, \ldots, (n-2)\} \) and \( V_2 = \{1, 3, \ldots, (n-1)\} \). In order to find out the Wiener polynomial of \( X_n \), we need to calculate \( d(u, v) \). For the procedure, let us take the condition \( u \in V_1 \) or \( u \in V_2 \). First we take \( u \in V_1 \)

**Claim 1:** \( d(u, v) = 2 \)

Let \( v \in V_1 \). Clearly, \( u \) and \( v \) are not adjacent. Then by Lemma 1.1, for \( u, v \in V_1 \), there exists a common neighbour. So \( d(u, v) = 2 \).

**Claim 2:** \( d(u, v) = 3 \)

Now, consider the case \( u \in V_1 \) and \( v \in V_2 \). It is understood that there exists \( \phi(n) \) neighbours of \( u \) in \( V_2 \). So we take \( V_2 = A \cup B \), where \( A = \{v \in V_2; uv \in E(X_n)\} \) and \( B = \{v \in V_2; uv \notin E(X_n)\} \). Obviously, for \( u \in V_1 \) and \( v \in A \), \( d(u, v) = 1 \). Let \( v \in B \). It follows that \( u \) and \( v \) are not adjacent. So take \( w \in A \subset V_2 \). Then \( uw \in E(X_n) \). But we can see that \( v \) and \( w \) are both odd. So there should exist a common neighbour \( x \) to \( v \) and \( w \) which results in the conclusion that \( d(u, v) = 3 \). The case of \( u \in V_2 \) is analogous to the case \( u \in V_1 \).

Thus it follows by definition of Wiener polynomial,

\[
W(X_n; q) = \sum_{\{u, v\}} q d(u, v) = \frac{n\phi(n)}{2} q + \frac{n(n-2)}{4} q^2 + \frac{n(n-2\phi(n))}{4} q^3.
\]
For \( n \) is odd but not prime, assume that \( p_1, p_2, \ldots, p_s \) are the different prime divisors of \( n \). Let \( n = p_1^{r_1} p_2^{r_2} \ldots p_s^{r_s} \), \( p_i \neq 2, 1 \leq i \leq s \). Since the factors in the expansion of \( F_n(a - b) \) in Lemma 1.1 are all positive, all the vertices are either adjacent or there exist a common neighbour to every pair of distinct vertices. This leads to the point that \( d(u, v) = 1 \) or 2. Hence again using the definition of Wiener polynomial, we reach the result that

\[
W(X_n; q) = \sum_{\{u, v\}} q^{d(u, v)} = \frac{n\phi(n)}{2} q^n + \frac{n(n - \phi(n) - 1)}{2} q^2.
\]

This completes the proof.

**HYPER-WIENER POLYNOMIAL OF UNITARY CAYLEY GRAPHS:**

**THEOREM 3.1:** If \( X_n \) is the Unitary Cayley graph, then the Hyper-Wiener polynomial of \( X_n \) is given by

\[
WW(X_n; q) = \begin{cases} 
\frac{n(n - 1)}{4} q^2, & \text{if } n \text{ is prime} \\
\frac{n\phi(n)}{2} q^2 + \frac{n(n - 2)}{4} q^6, & \text{if } n = 2^\alpha, \alpha > 1 \\
\frac{n\phi(n)}{2} q^2 + \frac{n(n - 2)}{4} q^6 + \frac{n(n - 2\phi(n))}{4} q^{12}, & \text{if } n \text{ is even and has an odd prime divisor} \\
\frac{n\phi(n)}{2} q^2 + \frac{n(n - \phi(n) - 1)}{2} q^6, & \text{if } n \text{ is odd but not prime}.
\end{cases}
\]

**PROOF:** The proof is quite direct from the proof of Theorem 2.1.

**CONCLUSION:**

In this paper, we direct our attention to the two polynomials, namely, Wiener and Hyper-Wiener polynomials. Also, we could form the result with the computation of Wiener and Hyper-Wiener polynomials of Unitary Cayley graphs.

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