Product Summability \((E, 1)(N, P_n)\) of Conjugate Series Of Fourier Series

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ABSTRACT

In the present study, some results on the product summability \((E, 1)(N, P_n)\) of Conjugate Fourier series have been established.

KEYWORDS: \((E, q)\) summability, \((N, P_n)\) summability, \((E, 1)(N, P_n)\) summability.

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INTRODUCTION

The study of Nörlund \((N, P_n)\) suumability of Fourier series and its allied series was first studied by Mears\(^1\) and then afterwards so many results deduced on the product summability of Nörlund means by a regular summability (i.e. in the form of \(X(N, P_n)\) or \((N, P_n)X\) where \(X\) is any regular summability). In the same context, Lal and Nigam\(^2\), Lal and Singh\(^3\), Prasad\(^4\), Sahney\(^5\), Sinha and Shrivastava\(^6\) and many researchers gave interesting results under different criteria & conditions. Therefore by inspiring this, under a very general condition, we have established some results on \((E,1)(N, P_n)\) summability of conjugate series of Fourier series. As a result, we see that the product operator gives better approximated value than individual linear operator.

Let \(\sum_{n=0}^{\infty} a_n\) be a given infinite series with the sequence of its partial sums \(\{S_n\}\). Let \(\{p_n\}\) be any sequence of constants, real or complex, such that
\[
P_n = p_0 + p_1 + p_2 + \cdots + p_n
\]
\[
P_{-1} = p_{-1} = 0
\]
Therefore,

The sequence-to-sequence transformation is given by
\[
t_n = \frac{1}{P_n} \sum_{k=0}^{n} P_{n-k} s_k
\]
defines the sequence \(\{t_n\}\) of Nörlund means of the sequence \(\{S_n\}\), as generated by the sequence of coefficients \(\{p_n\}\).

The series \(\sum_{n=0}^{\infty} a_n\) is said to be \((N, P_n)\) summable to the sum \(s\) if \(\lim_{n \to \infty} t_n\) exists and is equal to \(s\).

The necessary and sufficient condition for the regularity of \((N, P_n)\) method is
\[
\frac{P_n}{P_{n-1}} \to 0, \quad \text{as } n \to \infty
\]
Let,
\[
E^1_n = \frac{1}{2^n \binom{n}{k}} \sum_{k=0}^{n} \binom{n}{k} s_k
\]
If \(E^1_n \to s\), as \(n \to \infty\) then \(\sum_{n=0}^{\infty} a_n\) is said to be summable \(s\) by Euler means. Hardy\(^7\)

On superimposing \((E,1)\) transform on \((N, P_n)\) transform, we have the product \((E,1)(N, P_n)\) transform \(t_n^{EN}\) of the \(n^{th}\) partial series \(S_n\) of the series \(\sum_{n=0}^{\infty} a_n\) which is given by
then, the infinite series \( \sum_{n=0}^{\infty} a_n \) is said to be \((E,1)(N,P_n)\) summable to the sum \(s\), if \( t_n^{EN} \to s \) as \( n \to \infty \) i.e. the limit exist.

Let, \( f(t) \) be a periodic function with period \( 2\pi \) and Lebesgue-integrable over the interval \((-\pi, \pi)\). Then the Fourier series associated with \( f \) at any point \( t \) is defined by

\[
 f(t) \sim \sum_{n=1}^{\infty} \left( a_n \cos nt + b_n \sin nt \right) = \sum_{n=1}^{\infty} A_n(t)
\]

Then the conjugate series of (1.1) is

\[
 \sum_{n=1}^{\infty} \left( b_n \cos nt - a_n \sin nt \right) = \sum_{n=1}^{\infty} B_n(t)
\]

We use the following notations throughout this paper

\[
 \psi(t) = \frac{1}{2} [f(x + t) - f(x - t)]
\]

and

\[
 \bar{K}_n(t) = \frac{1}{2n} \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{p_k} \sum_{v=0}^{k} p_{k-v} S_v \cos \left( v \frac{t}{2} \right) \right\} \frac{\sin \left( \frac{t}{2} \right)}{\sin \left( \frac{t}{2} \right)}
\]

**KNOWN RESULTS**

Recently, Sinha and Shrivastava\(^6\) have discussed the almost \((E,q)(N,P_n)\) summability of Fourier Series by proving the following

**Theorem A.** If \( f \) is a \( 2\pi \) periodic function of class \( L^\alpha \) then the degree of approximation by the product \((E,q)(N,P_n)\) summability mean on its Fourier series (1.1) is given by

\[
 \| \tau_n - f \|_\infty = o \left( \frac{1}{(n+1)^\alpha} \right) \quad 0 < \alpha < 1
\]

where, \( \tau_n \) is defined as

\[
 \tau_n = \left\{ \frac{1}{(1 + q)^n} \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \left\{ \frac{1}{p_k} \sum_{v=0}^{k} p_{k-v} S_v \right\} \right\}
\]

Further, Prabhakar and Saxena\(^8\), have obtained an analogous result by generalised theorem A for \((E,1)(N,P_n)\) summability of Fourier series under different condition and criteria. The Theorems are as follows
**Theorem B.** Let \( \{c_n\} \) be a non-negative, monotonic, non-increasing sequence of real constants such that

\[
C_n = \sum_{v=1}^{n} c_v \to \infty, \text{ as } n \to \infty
\]

If

\[
\Phi(t) = \int_{0}^{t} |\phi(u)| \, du = o \left[ \frac{t}{\alpha \left( \frac{1}{t} \right) C_t} \right] \text{ as } t \to +0
\]  

(2.2)

where, \( \alpha(t) \) is a positive, monotonic and non-increasing function of \( t \) and \( \log(n + 1) = O[\{\alpha(n + 1)\} C_{n+1}] \), as \( n \to \infty \)

(2.3)

then the Fourier series (1.1) is \((E, 1)(N, P_n)\) summable to zero at point \( x \).

**MAIN RESULT**

With this point of view, we here prove the following theorems.

**Theorem 1.** Let \( \{c_n\} \) be a non-negative, monotonic, non-increasing sequence of real constants such that

\[
C_n = \sum_{v=0}^{n} c_v \to \infty \text{ as } n \to \infty
\]

If

\[
\Psi(t) = \int_{0}^{t} |\psi(u)| \, du = o \left[ \frac{t}{\alpha \left( \frac{1}{t} \right) C_t} \right] \text{ as } t \to +0
\]  

(3.1)

where, \( \alpha(t) \) is a positive, monotonic and non-increasing function of \( t \) and

\[
\log(n + 1) = O[\{\alpha(n + 1)\} C_{n+1}] \text{, as } n \to \infty
\]

(3.2)

then the conjugate Fourier series (1.2) is \((E, 1)(N, P_n)\) summable to

\[
\hat{f}(x) = -\frac{1}{2\pi} \int_{0}^{2\pi} \psi(t) \cot \left( \frac{t}{2} \right) dt
\]

at every pt, where this integral exists.

**Theorem 2:** Let \( \{c_n\} \) be a positive, monotonic, non-increasing sequence of real constants such that

\[
C_n = \sum_{v=0}^{n} c_v \to \infty \text{ as } n \to \infty
\]

If
\[ \Psi(t) = \int_0^t |\psi(u)| \, du = o\left(\frac{t}{\log \left(\frac{1}{t}\right)}\right), \quad \text{as } t \to +0 \]  

(3.3)

then the conjugate Fourier series (1.2) is \((E,1)(N,P_n)\) summable to

\[ \hat{f}(x) = -\frac{1}{2\pi} \int_0^{2\pi} \psi(t) \cot \left(\frac{t}{2}\right) \, dt \]

at every pt, where this integral exists.

To prove the following Theorems, we require the following lemmas.

**LEMMAS**

**Lemma 4.1**

For \(0 \leq t \leq \frac{1}{n+1}\), \(|\tilde{K}_n(t)| = O\left(\frac{1}{t}\right)\)

**Proof.**

\[
|\tilde{K}_n(t)| = \frac{1}{2^{n+1}t} \left| \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{\Pi_k} \sum_{\nu=0}^{k} P_{k-\nu} \frac{\cos \left(\nu + \frac{1}{2}\right) t}{\sin \left(\frac{t}{2}\right)} \right) \right|
\]

\[ \leq \frac{1}{2^{n+1}t} \left[ \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{\Pi_k} \sum_{\nu=0}^{k} P_{k-\nu} \left| \frac{\cos \left(\nu + \frac{1}{2}\right) t}{\sin \left(\frac{t}{2}\right)} \right| \right) \right] \]

\[ \leq \frac{1}{2^{n+1}t} \left[ \sum_{k=0}^{n} \binom{n}{k} \frac{1}{\Pi_k} \sum_{\nu=0}^{k} P_{k-\nu} \right] \]

\[ = \frac{(2n+1)}{2^{n+1}t} \cdot 2^n \]

\[ = O\left(\frac{1}{t}\right) \]

This completes the proof of Lemma 3.1

**Lemma 4.2**

For \(\frac{1}{n+1} \leq t \leq \pi\), \(|\tilde{K}_n(t)| = O\left(\frac{1}{t}\right)\)
**Proof.**

\[
|\tilde{K}_n(t)| = \frac{1}{2^{n+1} \pi} \left| \sum_{k=0}^{n} \left( \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} \frac{\cos \left( v + \frac{1}{2} \right) t}{\sin \left( \frac{t}{2} \right)} \right\} \right) \right| \\
\leq \frac{1}{2^{n+1} t} \left| \sum_{k=0}^{n} \left( \binom{n}{k} \right) \text{Re} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} \frac{p_{k-v} e^{(v+\frac{1}{2}) t}}{e^{\frac{t}{2}}} \right\} \right| \\
\leq \frac{1}{2^{n+1} t} \left| \sum_{k=0}^{n} \left( \binom{n}{k} \right) \text{Re} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} \frac{p_{k-v} e^{i \nu v t}}{e^{\frac{t}{2}}} \right\} \right| \\
\leq \frac{1}{2^{n+1} t} \left| \sum_{k=0}^{n} \left( \binom{n}{k} \right) \text{Re} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_{k-v} e^{i \nu v t} \right\} \right| \\
= |K_1| + |K_2|
\]

\[
|K_1| \leq \frac{1}{2^{n+1} t} \left| \sum_{k=0}^{n-1} \left( \binom{n}{k} \right) \text{Re} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_{k-v} e^{i \nu v t} \right\} \right| \\
\leq \frac{1}{2^{n+1} t} \left| \sum_{k=0}^{n-1} \left( \binom{n}{k} \right) \text{Re} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_{k-v} e^{i \nu v t} \right\} \right| \\
\leq \frac{1}{2^{n+1} t} \left| \sum_{k=0}^{n-1} \left( \binom{n}{k} \right) \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_{k-v} \right\} \right| \\
\leq \frac{1}{2^{n+1} t} \left| \sum_{k=0}^{n-1} \left( \binom{n}{k} \right) \right| \\
= O \left( \frac{1}{t} \right)
\]

Now considering second term and using Abel’s lemma

\[
|K_2| \leq \frac{1}{2^{n+1} t} \left| \sum_{k=0}^{n} \left( \binom{n}{k} \right) \text{Re} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_{k-v} e^{i \nu v t} \right\} \right| \\
\leq \frac{1}{2^{n+1} t} \left| \sum_{k=0}^{n} \left( \binom{n}{k} \right) \frac{1}{P_k} \max_{0 \leq m \leq k} \left| \sum_{v=0}^{k} p_{k-v} e^{i \nu v t} \right| \right| \\
= O \left( \frac{1}{t} \right)
\]

This completes the proof of Lemma 3.2 Similarly,

**Lemma 4.3**

For \(0 \leq t \leq \frac{1}{n}\),
\[|\tilde{K}_n(t)| = O\left(\frac{1}{t}\right)\]

**Lemma 4.4**

For \(\frac{1}{n} \leq t \leq \pi\),
\[|\tilde{K}_n(t)| = O\left(\frac{1}{t}\right)\]

**PROOF**

**Proof of Theorem 1:**

Let, \(\tilde{s}_n\) denote the partial sum of conjugate Fourier series (1.2) then following Zygmund, we have

\[
\tilde{s}_n - \tilde{f}(x) = \frac{1}{2\pi} \int_0^\pi \psi(t) \frac{\cos\left(\frac{n+1}{2} t\right)}{\sin \frac{t}{2}} dt
\]

Therefore the \((E,1)(N, P)\) transform of \(\tilde{s}_n(x)\) is given by

\[
\tilde{t}^\text{EN}_n - \tilde{f}(x) = \frac{1}{2^n+1\pi} \int_0^\pi \psi(t) \sum_{k=0}^n \left(\frac{n}{k}\right) \left\{ \frac{1}{P_k} \sum_{\nu=0}^k p_{k-\nu} \frac{\cos\left(\frac{\nu+1}{2} t\right)}{\sin \frac{t}{2}} \right\} dt
\]

\[
= \int_0^\pi \psi(t) |\tilde{K}_n(t)| dt
\]

For \(0 < \delta < \pi\), we have

\[
\int_0^\pi \psi(t) |\tilde{K}_n(t)| dt = \int_0^{1/n+1} \psi(t) |\tilde{K}_n(t)| dt + \int_{1/n+1}^\delta \psi(t) |\tilde{K}_n(t)| dt + \int_\delta^\pi \psi(t) |\tilde{K}_n(t)| dt
\]

\[
= I_1 + I_2 + I_3 \quad \text{(say)} \quad (5.1)
\]

Now, by applying (3.1), (3.2) and (4.1), we have

\[
|I_1| \leq \int_0^{1/n+1} |\psi(t)||\tilde{K}_n(t)| dt
\]

\[
= O \int_0^{1/n+1} \frac{1}{t} |\psi(t)| dt
\]

\[
= O(n+1) \int_0^{1/n+1} |\psi(t)| dt
\]

\[
= O(n+1) \left[ o\left(\frac{1}{(n+1)\alpha(n+1)C_{n+1}}\right) \right]
\]

\[
= o\left(\frac{1}{\log(n+1)}\right)
\]

\[
= o(1), \quad \text{as } n \to \infty \quad (5.2)
\]

From condition (3.1), (3.2) and (4.2), we have
\(|I_2| \leq \int_{1/n+1}^{\delta} |\psi(t)||\tilde{K_n}(t)|dt

= O\left[\int_{1/n+1}^{\delta} |\psi(t)|\left(\frac{1}{t}\right) dt\right]

= O\left[\frac{1}{t}\psi(t)\right]_{1/n+1}^{\delta} + \int_{1/n+1}^{\delta} \frac{1}{t^2} \psi(t) dt

= O\left[\left(\frac{1}{\alpha(t)}\right)_{1/n+1}^{\delta} + \int_{1/n+1}^{\delta} \left(\frac{1}{\alpha(t)}\right) dt\right]

Putting \(\frac{1}{t} = u\) in second term

= O\left[\left(\frac{1}{\log(n+1)}\right) + \int_{1/\delta}^{n+1} \left(\frac{1}{\log(u)}\right) du\right]

= o(1) + o(1), \quad \text{as } n \to \infty

= o(1), \quad \text{as } n \to \infty \quad (5.3)

By Riemann-Lebesgue lemma & by regularity condition of the method of summability, we have

\(|I_3| \leq \int_{\delta}^{\pi} |\psi(t)||\tilde{K_n}(t)|dt

= o(1), \quad \text{as } n \to \infty \quad (5.4)

Combining (5.1), (5.2), (5.3) and (5.4), we have

\(I_1 + I_2 + I_3 = o(1)\)

Hence we proved that

\(\tilde{t}_n^{E,1}(N,P_n) - \tilde{f}(x) = o(1), \quad \text{as } n \to \infty\)

This completes the proof of Theorem 1.

\textbf{Proof of Theorem 2:}

For \(0 < \delta < \pi\),

\(\tilde{t}_n^{E,1}(N,P_n) - \tilde{f}(x) = \int_0^\pi \psi(t)\tilde{K}_n(t) dt\)
\[
\begin{align*}
&= \int_0^{1/n} \psi(t)\bar{K}_n(t)\,dt + \int_{1/n}^{\delta} \psi(t)\bar{K}_n(t)\,dt + \int_{\delta}^{\pi} \psi(t)\bar{K}_n(t)\,dt \\
&= J_1 + J_2 + J_3 \quad \text{(say)} \quad (5.5)
\end{align*}
\]

On applying (3.3) and (4.3), we have
\[
|J_1| = \int_0^{1/n} \left| \psi(t) \right| \left| \bar{K}_n(t) \right| \,dt \\
= O\left[ \int_0^{1/n} \frac{1}{t} \left| \psi(t) \right| \,dt \right] \\
= O(n) \left[ \int_{1/n}^{1} \left| \psi(t) \right| \,dt \right] \\
= O\left( \frac{1}{\log(n)} \right) \\
= o(1), \quad \text{as } n \to \infty \quad (5.6)
\]

From (3.3) and (4.4), we have
\[
|J_2| = \int_{1/n}^{\delta} \left| \psi(t) \right| \left| \bar{K}_n(t) \right| \,dt \\
= O\left[ \int_{1/n}^{\delta} \frac{1}{t} \left| \psi(t) \right| \,dt \right] \\
= O\left[ \frac{1}{\bar{t}} \psi(t) \right]_{1/n}^{\delta} + \int_{1/n}^{\delta} \frac{1}{t^2} \psi(t) \,dt \\
= O\left[ \frac{1}{\log(\frac{\bar{t}}{n})} \right]_{1/n}^{\delta} + \int_{1/n}^{\delta} \frac{1}{t \log(\frac{\bar{t}}{n})} \,dt \\
= o\left( \frac{1}{\log(\bar{t} \log(n))} \right) + o(1) \left\{ -\log \log \left( \frac{1}{\bar{t}} \right) \right\}_{1/n}^{\delta} \\
= o(1) + o(1), \quad \text{as } n \to \infty \\
= o(1), \quad \text{as } n \to \infty \quad (5.7)
\]

Finally,

By using Riemann-Lebesgue theorem and regularity condition of summability, we have
\[ |J_3| = \int_{\delta}^{\pi} |\psi(t)||\tilde{K}_n(t)|dt = o(1) , \text{ as } n \to \infty \tag{5.8} \]

Combining (5.5), (5.6), (5.7) and (5.8) we have
\[ \tau_n^{(E,1),(N,P_n)} - \tilde{f}(x) = o(1) , \text{ as } n \to \infty \]

This completes the proof of Theorem 2.

CONCLUSION

Several results concerning the product summability of Nörlund-Euler means have been reviewed with different criteria and conditions. In future, by applying more conditions we can rectify the errors and its application in the field of Fourier analysis.

REFERENCES