Vertex Even Mean Labeling of New Families of Graphs

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ABSTRACT

The main aim of this paper is to identify some cycle related graphs which admit vertex even mean labeling. We prove that every cycle \( C_n \) (\( n \geq 6 \)) with parallel chords is vertex even mean graph. Using graph operations on cycles with parallel chords we have obtained new families of graphs namely chain of even cycles with parallel chords, crown with parallel chords, subdivided cycle with parallel chords, two copies of odd cycles sharing a common edge, two copies of odd cycles sharing a common vertex which are vertex even mean graphs.

KEYWORDS: Mean labeling, Vertex Even Mean labeling, Cycles with parallel chords, Subdivided Cycle with parallel chords, Chain of cycles

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I. INTRODUCTION

In Graph theory labeling of a graph is obtained by assigning integer values to the vertices or edges of a graph under some conditions. \( \beta \)-valuation introduced by Rosa was the origin of graph labelling which was later renamed by Golomb as graceful labeling. In this paper, a finite simple and undirected graph \( G = (V, E) \) with \( p \) vertices and \( q \) edges is considered. Many kinds of labeling and its variations have been introduced by researchers over the past five decades. Mean labeling was introduced by Somasundaram and et al. Several graphs that are proved to be mean graphs are seen in. Various types of mean labeling have been introduced since then. N. Revathi introduced a kind of mean labeling namely vertex even mean labeling and proved that Umbrella graph, \( K_1 + C_n \) are vertex even mean graphs. Vertex even mean labeling of various graphs are discussed in. Gallian gives a detailed survey on graph labeling. Several labelings on cycles with zigzag chords, cycles with parallel chords are seen in. This paper is focused on vertex even mean labeling of cycles with parallel chords and also new classes of graphs obtained from cycles with parallel chords using graph operations.

Definition 1.1:

A graph \( G \) with \( |V(G)|=p \), and \( |E(G)|=q \) is said to be a mean graph if there is an injection \( f : V(G) \rightarrow \{0,1,2,...,q\} \) such that when each edge \( xy \) is labeled with \( \frac{f(x)+f(y)}{2} \) if \( f(x)+f(y) \) is even, and \( \frac{f(x)+f(y)+1}{2} \) if \( f(x)+f(y) \) is odd, the resulting edge labels are distinct.

Definition 1.2:

A Graph \( G=(V, E) \) with \( |V(G)|=p \), and \( |E(G)|=q \) is said to be vertex even mean graph if there exist an injection \( f : V(G) \rightarrow \{2,4,..,2q\} \) such that the induced mapping \( f^*:E(G)\rightarrow I \{\text{set of positive integers}\} \) defined by \( f^*(uv) = \frac{f(u)+f(v)}{2} \) are distinct for each edge \( uv \). Such a function \( f \) is called a vertex even mean labeling.

Definition 1.3:

A cycle with parallel chords is defined as a graph \( G \) obtained from a cycle \( C_n \) (\( n \geq 6 \)) with consecutive vertices \( v_0,v_1,...,v_{n-1},v_0 \) by adding the chords \( v_1 v_{n-1}, v_2 v_{n-2}, ... v_\alpha v_\beta \) where \( \alpha = \left\lfloor \frac{n}{2} \right\rfloor -1 , \beta = \left\lfloor \frac{n}{2} \right\rfloor + 1 \) if \( n \) is even and \( \beta = \left\lfloor \frac{n}{2} \right\rfloor + 2 \) if \( n \) is odd. Then \( G \) has \( n \) vertices and \( M \) edges where \( M = (3n-3) / 2 \) if \( n \) is odd and \( M = (3n-2) / 2 \) if \( n \) is even. Here chord is an edge connecting two otherwise non adjacent vertices of the cycle \( C_n \).
**Definition 1.4:**
Consider $n$ copies of cycle $C_{2m}$. Chain of Cycles is the graph obtained by identifying $v_{i,m}$ with $v_{i+1,m}$ for $i = 1, 2, \ldots, n-1$ where $v_{i,1}, v_{i,2} \ldots v_{i,2m}$ are the successive vertices of $n$ copies of $C_{2m}$.

**Definition 1.5:**
Subdivision of a graph is a graph obtained by subdividing each edge of the graph with a vertex exactly once.

**Definition 1.6:**
We define Subdivided cycle with parallel chords as a graph obtained by subdividing each edge of cycle only with a vertex.

**II. MAIN RESULTS**

**Theorem 2.1:** Every cycle $C_n$ ($n \geq 6$) with parallel chords is a vertex even mean graph

**Proof:** Let $G$ be the cycle $C_n$ with parallel chords. Then by definition 1.3 $G$ has $n$ vertices and $M$ edges. The $n$ vertices of $G$ are $v_0, v_1, \ldots, v_{n-1}$. Define the vertex labelling $f : V(G) \rightarrow \{2, 4, \ldots, 2M\}$ as follows for the two cases depending on $n$.

- $f(v_i) = 4i$, $1 \leq i \leq n/2$ if $n$ is even and $1 \leq i \leq (n-1)/2$ if $n$ is odd
- $f(v_{n-i}) = 4i + 2$, $0 \leq i \leq n/2 - 1$ if $n$ is even and $0 \leq i \leq (n-1)/2$ if $n$ is odd

From the above it is clear that all the vertices labeled are distinct.

$E(G) = E_1 \cup E_2 \cup E_3 \cup E_4$ is the edge set where

- $E_1 = \{v_iv_{i+1}, 1 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 1 \leq i \leq (n-3)/2 \text{ if } n \text{ is odd}\}$
- $E_2 = \{v_{i+1}v_{n-i+1}, 0 \leq i \leq n/2 - 2 \text{ if } n \text{ is even and } 0 \leq i \leq (n-3)/2 \text{ if } n \text{ is odd}\}$
- $E_3 = \{v_iv_{n-i}, 1 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd}\}$
- $E_4 = \{v_0v_1, v_{n/2}v_{n/2+1} \text{ if } n \text{ is even and } v_0v_1 \text{ if } n \text{ is odd}\}$.

Define the induced function $f^* : E(G) \rightarrow I$ (Set of Positive integers) as

- $f^*(v_iv_{i+1}) = 4i + 2$, $1 \leq i \leq n/2 - 1$ if $n$ is even and $1 \leq i \leq (n-3)/2$ if $n$ is odd
- $f^*(v_{i+1}v_{n-i+1}) = 4i + 4$, $0 \leq i \leq n/2 - 2$ if $n$ is even and $0 \leq i \leq (n-3)/2 \text{ if } n \text{ is odd}$
- $f^*(v_iv_{n-i}) = 4i + 1$, $1 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd}$
- $f^*(v_0v_1) = 3$
- $f^*(v_{n/2}v_{n/2+1}) = 2n - 1$

It is clear from the above labeling that the edge labels of $E_1$, $E_2$ are even and that of $E_3$ is odd. To show that the edge labels of $E_1$, $E_2$ are distinct we will assume on the contrary that they are same.

For the edges of $E_1$ and $E_2$ :

- if $i \neq j$, $1 \leq i \leq n/2 - 1$ and $0 \leq j \leq n/2 - 2$
let \( f^* (v_{i+1}) = f^* (v_{n-j+1}) \) Then
\[
4i + 2 = 4j + 4 \Rightarrow 2(i-j) = 1
\]
Here the left hand side is an even integer whereas the right hand side is an odd integer, a contradiction. Hence the edge labels of \( E_1 \) and \( E_2 \) are distinct. Thus all the edges of \( G \) have distinct labels and \( G \) is a vertex even mean graph. An illustration is given in Figure 1 and Figure 2.

**Theorem 2.2**: Chain of even cycles \( C_n \) (\( n \geq 6 \)) with parallel chords is a vertex even mean graph.

**Proof**: Consider \( s \) copies of even cycle \( C_n \) with parallel chords. By definition 1.4 for \( j = 1, 2, 3, \ldots, s \) let \( v_{0}, v_{1}, \ldots, v_{n-1} \) be the \( n \) vertices of \( j^{th} \) copy of cycle \( C_n \) with parallel chords. Chain of cycles \( C_n, s \) is the graph obtained from \( s \) copies of cycle \( C_n \) with parallel chords by identifying \( v_{n/2}, v_{0}, v_{1}, \ldots, v_{n-1} \) for \( j = 1, 2, \ldots, s \). Let this graph be denoted by \( G \). Then \( G \) has \( s(n-1) + 1 \) vertices and \( sM \) edges.

Define the vertex labeling \( f : V(G) \rightarrow \{2, 4, \ldots, 2sM\} \) as follows:

For \( j = 1, 2, \ldots, s \)
\[
f(v_{i,j}) = 4i + 2(n-1)(j-1), \quad 1 \leq i \leq n/2 \]
\[
f(v_{n-i,j}) = 4i + 2 + 2(n-1)(j-1), \quad 0 \leq i \leq n/2 - 1
\]
From the above it is clear that all the vertices labeled are distinct.

\( E(G) = E_1 \cup E_2 \cup E_3 \cup E_4 \) is the edge set where
\[
E_1 = \{v_{i,j}v_{i+1,j} : 1 \leq i \leq n/2 - 1 \text{ and } 1 \leq j \leq s\}
\]
\[
E_2 = \{v_{n-i,j}v_{n-(i+1), j} : 0 \leq i \leq n/2 - 2 \text{ and } 1 \leq j \leq s\}
\]
\[
E_3 = \{v_{i,j}v_{n-i-j} : 1 \leq i \leq n/2 - 1 \text{ and } 1 \leq j \leq s\}
\]
\[
E_4 = \{v_{0,j}v_{1,j}v_{n/2,j}v_{n/2+1,j} : 1 \leq j \leq s\}
\]
Define the induced function \( f^* : E(G) \rightarrow I \) (Set of Positive integers) as follows:
For \( j = 1,2, \ldots, s \)
\[
\begin{align*}
 f^*(v_{ij}v_{i+1,j}) &= 4i + 2 + 2(j-1)(n-1), & 1 \leq i \leq n/2 - 1 \\
 f^*(v_{n,i}v_{n(i+1),j}) &= 4i + 4 + 2(j-1)(n-1), & 0 \leq i \leq n/2 - 2 \\
 f^*(v_{i,j}v_{i+1,n-j}) &= 4i + 1 + 2(j-1)(n-1), & 1 \leq i \leq n/2 - 1 \\
 f^*(v_{0,j}v_{1,j}) &= 3 + 2(j-1) (n-1) \\
 f^*(v_{n/2,j}v_{n/2+1,j}) &= 2n - 1 + 2(j-1) (n-1)
\end{align*}
\]
From the above labeling it is evident that the edge labels of \( E_2 \), \( E_3 \), \( E_4 \) and \( E_5 \) are in an increasing sequence as \( i \) increases and are distinct. Also that the edge labels of \( E_4 \) are distinct as \( 2n - 1 > 3 \) for \( n \geq 6 \).

Hence \( G \) is a vertex even mean graph. An illustration for this case is given in Figure 3.

**Theorem 2.3:** Every crown \( C_n \circ K_1 \) \((n \geq 6)\) with parallel chords is a vertex even mean graph

**Proof:** A cycle \( C_n \) with parallel chords is considered. By attaching a pendant edge at each vertex of cycle \( C_n \) with parallel chords, a Crown \( C_n \circ K_1 \) with parallel chords is obtained. Let this graph be \( G' \).

The vertices of cycle \( C_n \) are denoted by \( v_0, v_1, \ldots, v_{n-1} \) and let \( v_0', v_1', \ldots, v_{n-1}' \) be the pendant vertices of \( G' \). Then by definition 1.4, \( G' \) has \( 2n \) vertices and \( M + n \) edges where \( M = (3n-3)/2 \) if \( n \) is odd and \( M = (3n-2)/2 \) if \( n \) is even. Define the vertex labelling \( f : V(G') \to \{2,4,\ldots,2(M+n)\} \) as follows for the two cases depending on \( n \).

\[
\begin{align*}
 f(v_i) &= 8i - 2, & 1 \leq i \leq n/2 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd} \\
 f(v_{n-i}) &= 8i + 2, & 0 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 0 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd} \\
 f(v_0') &= \begin{cases} 4n - 4 & \text{if } n \text{ is even} \\ 4n & \text{if } n \text{ is odd} \end{cases} \\
 f(v_i') &= 8i - 4, & 1 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd} \\
 f(v_{n-i}') &= 8i, & 1 \leq i \leq n/2 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd}
\end{align*}
\]

From the above it is seen that all the vertices are labeled and are distinct.

\( E(G) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \) is the edge set where

\[
\begin{align*}
 E_1 &= \{v_i v_{i+1}, 1 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 1 \leq i \leq (n-3)/2 \text{ if } n \text{ is odd} \} \\
 E_2 &= \{v_{n-i} v_{n-(i+1)}, 0 \leq i \leq n/2 - 2 \text{ if } n \text{ is even and } 0 \leq i \leq (n-3)/2 \text{ if } n \text{ is odd} \}
\end{align*}
\]
\[ E_3 = \{ v_i v_{n-i} \mid 1 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd} \} \]
\[ E_4 = \{ v_0 v_0', v_0 v_1 \} \]
\[ E_1^* = \{ v_i v_i', 1 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd} \} \]
\[ E_2^* = \{ v_{n-i} v_{n-i}', 1 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd} \} \]
\[ E_3^* = \{ v_{n/2} v_{n/2-1}' \text{ if } n \text{ is even} \} \]

Define the induced function \( f^* : E(G^*) \rightarrow I \) (Set of Positive integers) as
\[ f^* (v_i v_{i+1}) = 8i + 2, \quad 1 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 1 \leq i \leq (n-3)/2 \text{ if } n \text{ is odd} \]
\[ f^* (v_n v_{n-(i+1)}) = 8i + 6, \quad 0 \leq i \leq n/2 - 2 \text{ if } n \text{ is even and } 0 \leq i \leq (n-3)/2 \text{ if } n \text{ is odd} \]
\[ f^* (v_i v_{n-i}) = 8i, \quad 1 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd} \]
\[ f^* (v_0 v_0') = 2n - 1 \quad \text{if } n \text{ is even} \]
\[ f^* (v_0 v_1') = 2n + 1 \quad \text{if } n \text{ is odd} \]
\[ f^* (v_i v_1) = 4 \]
\[ f^* (v_i v_i') = 8i - 3, \quad 1 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd} \]
\[ f^* (v_n v_{n-1}') = 8i + 1, \quad 1 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd} \]

If \( n \) is even,
\[ f^* (v_{n/2} v_{n/2 + 1}) = 4n - 4 \]
\[ f^* (v_{n/2} v_{n/2}') = 4n - 1. \]

From the above labeling it is observed that all the edge labels are distinct and \( G^* \) is a vertex even mean graph. An illustration for this case is given in Figure 4 & Figure 5.
Theorem 2.4: Subdivided Cycle $C_n$ ($n \geq 6$) with parallel chords is a vertex even mean graph

Proof: By definition 1.6, let $G$ be the subdivided Cycle $C_n$ with parallel chords. The vertices of cycle $C_n$ are denoted by $v_0, v_1, \ldots, v_{n-1}$. Let the newly added vertices be $d_0, d_1, \ldots, d_{\frac{n-1}{2}}$ of subdivided edges $v_0v_1, v_1v_2, \ldots, v_{n-1}v_n$. Then $G$ has $2n$ vertices and $M + n$ edges.

Define the vertex labeling $f : V(G) \rightarrow \{2, 4, \ldots, 2(M + n)\}$ as

$$f(v_i) = 8i,$$  

$1 \leq i \leq n/2$ if $n$ is even and $1 \leq i \leq (n-1)/2$ if $n$ is odd

$$f(v_{n-i}) = 8i + 2,$$  

$0 \leq i \leq n/2 - 1$ if $n$ is even and $1 \leq i \leq (n-1)/2$ if $n$ is odd

$$f(d_i) = 8i + 4,$$  

$0 \leq i \leq n/2 - 1$ if $n$ is even and $1 \leq i \leq (n-1)/2$ if $n$ is odd

$$f(d_{n-i}) = 8i - 2,$$  

$1 \leq i \leq n/2$ if $n$ is even and $1 \leq i \leq (n-1)/2$ if $n$ is odd

It is seen from the above that all the vertices labeled are distinct.

The edge set $E(G) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6$ is given by

$E_1 = \{v_i d_i, 1 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd}\}$

$E_2 = \{v_i d_{i+1}, 1 \leq i \leq n/2 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd}\}$

$E_3 = \{v_{n-i} d_{n-i}, 1 \leq i \leq n/2 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd}\}$

$E_4 = \{v_i d_{n-i}, 1 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd}\}$

$E_5 = \{v_i v_{n-i}, 1 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 1 \leq i \leq (n-3)/2 \text{ if } n \text{ is odd}\}$

$E_6 = \{v_0 d_0 \text{ and } E_7 = \{v_n d_{n/2} \text{ if } n \text{ is even and } v_{(n+1)/2} d_{(n-1)/2} \text{ if } n \text{ is odd}\}$

Define the induced function $f^* : E(G) \rightarrow \mathbb{I}$ (Set of Positive integers) as

$$f^*(v_i d_i) = 8i + 2,$$  

$1 \leq i \leq n/2 - 1$ if $n$ is even and $1 \leq i \leq (n-1)/2$ if $n$ is odd

$$f^*(v_i d_{i+1}) = 8i - 2,$$  

$1 \leq i \leq n/2$ if $n$ is even and $1 \leq i \leq (n-1)/2$ if $n$ is odd
\[ f^* (v_{n(i-1)} d_{n,i}) = 8i - 4, \ 1 \leq i \leq n/2 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd} \]

\[ f^* (v_i d_i) = 8i, \quad 1 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd} \]

\[ f^* (v_i d_i) = 8i + 1, \quad 1 \leq i \leq n/2 - 1 \text{ if } n \text{ is even and } 1 \leq i \leq (n-1)/2 \text{ if } n \text{ is odd} \]

\[ f^* (v_0 d_0) = 3 \text{ and } f^* (v_{n/2} d_{n/2}) = 4n - 1 \text{ if } n \text{ is even} \]

\[ f^* (v_{(n+1)/2} d_{(n-1)/2}) = 4n - 1 \text{ if } n \text{ is odd} \]

It is clear from the above that all the edges receive distinct labels and \( G \) is a vertex even mean graph. An illustration for this case is given in Figure 6 & Figure 7.

**Theorem 2.5:** Two copies of odd cycle \( C_n \) \((n \geq 7)\) with parallel chords sharing a common edge is a vertex even mean graph

**Proof:** Consider two copies of odd cycle \( C_n \) with parallel chords. Let \( v_0, v_1, \ldots, v_{n-1} \) be the vertices of first copy of cycle \( C_n \) and let \( w_0, w_1, \ldots, w_{n-1} \) be the vertices of second copy of cycle \( C_n \) sharing a common edge \( v_{n-1} v_{n-2} = w_1 w_2 \). Let this graph be \( G \) having \( 2n - 2 \) vertices and \( 3n - 4 \) edges.

Define the vertex labeling \( f : V(G) \to \{2, 4, \ldots, 6n - 8\} \) as follows:

\[ f (v_i) = 4i, \quad 1 \leq i \leq (n-1)/2 \]

\[ f (v_{n-i}) = 4i + 2, \quad 0 \leq i \leq (n-1)/2 \]

\[ f (w_i) = 4n + 2i - 6, \quad 0 \leq i \leq (n-1)/2 \]

\[ f (w_{n-i}) = 5n + 2i - 7, \quad 3 \leq i \leq (n-1)/2 \]
All the vertices are labelled and are distinct.

\[ E(G) = E_1 U E_2 U E_3 U E_4 U E_5 U E_6 U E_7 U E_8 \]
is the edge set where

\[ E_1 = \{ v_i v_{i+1}, \ 1 \leq i \leq (n-3)/2 \} \]
\[ E_2 = \{ v_n i v_{n-(i+1)}, \ 0 \leq i \leq (n-3)/2 \} \]
\[ E_3 = \{ v_i v_{i+1}, \ 1 \leq i \leq (n-1)/2 \} \]
\[ E_4 = \{ w_i w_{i+1}, \ 3 \leq i \leq (n-3)/2 \} \]
\[ E_5 = \{ w_i v_{n-(i+1)}, \ 0 \leq i \leq (n-3)/2 \} \]
\[ E_6 = \{ w_i w_{n-i}, \ 3 \leq i \leq (n-1)/2 \} \]
\[ E_7 = \{ w_i w_{n-i}, \ i = 1,2 \} \]
\[ E_8 = \{ v_0 v_1, w_0 w_1, w_2 w_3 \} \]

Define the induced function \( f^* : E(G) \rightarrow \mathbb{I} \) (Set of Positive integers) as follows.

\[ f^*(v_i v_{i+1}) = 4i + 2, \quad 1 \leq i \leq (n-3)/2 \]
\[ f^*(v_n i v_{n-(i+1)}) = 4i + 4, \quad 0 \leq i \leq (n-3)/2 \]
\[ f^*(v_i v_{i+1}) = 4i + 1, \quad 1 \leq i \leq (n-1)/2 \]
\[ f^*(w_i w_{i+1}) = 5n + 2i - 6, \quad 3 \leq i \leq (n-3)/2 \]
\[ f^*(w_n i w_{n-(i+1)}) = 4n + 2i - 5, \quad 0 \leq i \leq (n-3)/2 \]
\[ f^*(w_i w_{n-i}) = (9n + 4i - 13)/2, \quad 3 \leq i \leq (n-1)/2 \]
\[ f^*(w_i w_{n-i}) = 2n + 3i - 2, \quad i = 1,2 \]
\[ f^*(v_0 v_1) = 3, \ f^*(w_0 w_1) = 2n, \ f^*(w_2 w_3) = (5n + 9)/2 \]

From the above it is clearly observed that all the edges receive distinct labels. Hence \( G \) is a vertex even mean graph. An illustration for this case is given in Figure 8.

**Theorem 2.6**: Two copies of odd cycle \( C_n \) (\( n \geq 7 \)) with parallel chords sharing a common vertex is a vertex even mean graph.
Proof: Consider two copies of odd cycle $C_n$ with parallel chords. Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of first copy of cycle $C_n$ and let $w_0, w_1, \ldots, w_{n-1}$ be the vertices of second copy of cycle $C_n$ sharing a common vertex $v_0$ which is same as $w_0$. Let this graph be denoted by $G$. Then $G$ has $2n - 1$ vertices and $3n - 3$ edges.

Define the vertex labeling $f: V(G) \rightarrow \{2, 4, \ldots, 6n - 6\}$ as

$$f(v_i) = 4i, \quad 1 \leq i \leq \frac{n-1}{2}$$

$$f(v_{n-i}) = 4i + 2, \quad 0 \leq i \leq \frac{n-1}{2}$$

$$f(w_i) = 4n + 4i - 6, \quad 1 \leq i \leq \frac{n-1}{2}$$

$$f(w_{n-i}) = 4n + 4i - 4, \quad 1 \leq i \leq \frac{n-1}{2}$$

It is seen from the above that all the vertices label are distinct.

The edge set $E(G) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7$ is given by

$$E_1 = \{v_1, v_{i+1}, 1 \leq i \leq (n-3)/2\}$$

$$E_2 = \{v_i, v_{n-(i+1)}, 0 \leq i \leq (n-3)/2\}$$

$$E_3 = \{v_i, v_{n-i}, 1 \leq i \leq (n-1)/2\}$$

$$E_4 = \{w_i, w_{i+1}, 1 \leq i \leq (n-3)/2\}$$

$$E_5 = \{w_i, w_{n-(i+1)}, 1 \leq i \leq (n-3)/2\}$$

$$E_6 = \{w_i, w_{n-i}, 1 \leq i \leq (n-1)/2\}$$

and $E_7 = \{v_0, v_1, w_1, v_0, w_{n-1}\}$

Define the induced function $f^*: E(G) \rightarrow I$ (Set of Positive integers) as

$$f^*(v_{i+1}) = 4i + 2, \quad 1 \leq i \leq (n-3)/2$$

$$f^*(v_{n-(i+1)}) = 4i + 4, \quad 0 \leq i \leq (n-3)/2$$

$$f^*(v_i, v_{n-i}) = 4i + 1, \quad 1 \leq i \leq (n-1)/2$$

$$f^*(w_i, w_{i+1}) = 4n + 4i - 4, \quad 1 \leq i \leq (n-3)/2$$

$$f^*(w_i, w_{n-(i+1)}) = 4n + 4i - 2, \quad 1 \leq i \leq (n-3)/2$$

$$f^*(w_i, w_{n-i}) = 4n + 4i - 5, \quad 1 \leq i \leq (n-1)/2$$

$$f^*(v_0, v_1) = 3,$$

$$f^*(v_0, w_1) = 2n$$

and

$$f^*(v_0, w_{n-1}) = 2n + 1$$

From the above it is evident that all the edge labels are distinct. Hence $G$ is a vertex even mean graph. An illustration is given in Figure 9.
CONCLUSION

In this paper we have obtained new families of graphs that are vertex even mean graphs using graph operations on cycles with parallel chords. In future we prove yet another labeling on similar graph families.

REFERENCES