Some common fixed point theorems for three mappings in Vector b-metric spaces

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ABSTRACT

In this paper we prove some common fixed point results for three mappings in vector b-metric space. Our results extend and improve some well-known results in literature. We also give an example to justify our results.

KEYWORDS: b-metric space, contraction mapping theorem, vector b-metric space, Rieszspace, weakly compatible.

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1. INTRODUCTION

Common fixed point theorems for three mappings in metric space were studied by Latpate et al. Similar results can be seen in Abbas et al, Arshad et al, Jungck and Rahimi et al. Further, these results were extended for vector metric space by Altun and Cevik. We extend some of the results of fixed point for three mappings defined on vector b-metric space which is a Riesz space valued metric space. Vector b-metric space was defined by Petre in 2014 by defining b-metric on vector metric space. We recall the basic concepts and definitions introduced by Altun and Cevik and Petre.

We follow notions and terminology by Aliprantis and Border, Luxemburg and Zannen for Riesz spaces.

A partially ordered set \((E, \leq)\) is a lattice if each pair of elements has a supremum and infimum. A real linear space \(E\) with an order relation \(\leq\) on \(E\) which is compatible with the algebraic structure of \(E\) is called an ordered linear space. Riesz space is an ordered vector space and at the same time a lattice also. Let \(E\) be a Riesz space with the positive cone \(E_+ = \{x \in E : x \geq 0\}\). For an element \(x \in E\), the absolute value \(|x|\), the positive part \(x^+\), the negative part \(x^-\) are defined as \(|x| = x \vee (-x), x^+ = x \vee 0, x^- = (-x) \vee 0\) respectively.

If every non–empty subset of \(E\) which is bounded above has a supremum, then \(E\) is called Dedekind complete or order complete. The Riesz space \(E\) is said to be Archimedean if \(\frac{1}{a} \downarrow 0\) holds for every \(a \in E_+\).

Let \(E\) be a Riesz space. A sequence \((b_n)\) is said to be order convergent or o–convergent to \(b\) if there is a sequence \((a_n)\) in \(E\) satisfying \(a_n \downarrow 0\) and \(|b_n - b| \leq a_n\) for all \(n\), written as \(b_n \xrightarrow{0} b\) or \(\lim o b_n = b\).

A sequence \((b_n)\) is said to be order Cauchy (o-Cauchy) if there exists a sequence \((a_n)\) in \(E\) such that \(a_n \downarrow 0\) and \(|b_n - b_{n+p}| \leq a_n\) holds for all \(n\) and \(p\).

A Riesz space \(E\) is said to be o-Cauchy complete if every o-Cauchy sequence is o-convergent.

**DEFINITION 1.1** \([10]\) : Let \(X\) be a non–empty set and \(E\) be a Riesz space. Then function \(d : X \times X \rightarrow E\) is said to be a vector metric (or \(E\)–metric) if it satisfies the following properties:

(a) \(d(x, y) = 0\) if and only if \(x = y\)

(b) \(d(x, y) \leq d(x, z) + d(y, z)\) for all \(x, y, z \in X\).
Also the triple \((X, d, E)\) is said to be a vector metric space. Vector metric space is a generalization of metric space. For arbitrary elements \(x, y, z, w\) of a vector metric space, the following statements are satisfied:

(i) \(0 \leq d(x, y)\)

(ii) \(d(x, y) = d(y, x)\)

(iii) \(|d(x, z) - d(y, z)| \leq d(x, y)\)

(iv) \(|d(x, z) - d(y, w)| \leq d(x, y) + d(z, w)\)

A sequence \((x_n)\) in a vector metric space \((X, d, E)\) vectorially converges (E-converges) to some \(x \in E\), written as \(x_n \xrightarrow{d,E} x\) if there is a sequence \((a_n)\) in \(E\) satisfying \(a_n \downarrow 0\) and \(d(x_n, x) \leq a_n\) for all \(n\).

A sequence \((x_n)\) is called E-cauchy sequence whenever there exists a sequence \((a_n)\) in \(E\) such that \(a_n \downarrow 0\) and \(d(x_n, x_{n+p}) \leq a_n\) holds for all \(n\) and \(p\).

A vector metric space \(X\) is called E-complete if each E-cauchy sequence in \(X\), E converges to a limit in \(X\).

For more detailed discussion regarding vector metric spaces we refer to \(^{6,8}\).

When \(E = R\), the concepts of vectorial convergence and metric convergence, E-cauchy sequence and Cauchy sequence in metric are same.

When also \(X = E\) and \(d\) is the absolute valued vector metric on \(X\), then the concept of vectorial convergence and convergence in order are the same.

**DEFINITION 1.2:** Let \(X\) be a non-empty set and let \(s \geq 1\) be a given real number. A function \(d : X \times X \to R^+\) is called a b-metric provided that, for all \(x, y, z \in X\)

(i) \(d(x, y) = 0\) if and only if \(x = y\)

(ii) \(d(x, y) = d(y, x)\)

(iii) \(d(x, z) \leq s[d(y, x) + d(y, z)]\)

A pair \((X, d)\) is called a b-metric space. It is clear from definition that b-metric space is an extension of usual metric space.

Several authors have investigated fixed point theorems on b-metric spaces, one can see \(^{11, 12}\).

Petre\(^7\) defined E-b-metric space or vector b-metric space as follows:

**DEFINITION 1.3\(^\text{[7]}\):** Let \(X\) be a nonempty set and \(s \geq 1\), A functional \(d : X \times X \to E_+\) is called an E-b-metric if for any \(x, y, z \in X\), the following conditions are satisfied:

(a) \(d(x, y) = 0\) if and only if \(x = y\)

(b) \(d(x, y) = d(y, x)\)

(c) \(d(x, z) \leq s[d(x, y) + d(y, z)]\)
The triple \((X, d, E)\) is called \(E\)-b-metric space.

**EXAMPLE 1.4:** Let \(d : [0,1] \times [0,1] \rightarrow \mathbb{R}^2\) defined by \(d(x,y) = (\alpha|x-y|^2, \beta|x-y|^2)\) then \((X,d,R^2)\) is \(E\)-b-metric space where \(\alpha, \beta > 0\).

**DEFINITION 1.5\cite{13}:** Let \(A\) and \(B\) be self maps of a set \(X\) if \(y = Ax = Bx\) for some \(x \in X\), then \(y\) is said to be a point of coincidence and \(x\) is said to be a coincidence point of \(A\) and \(B\). A pair of maps \(A\) and \(B\) is called weakly compatible pair if they commute at coincidence points\cite{8,11}.

**LEMMA 1.6 \cite{13}:** If \(E\) is a Riesz space and \(a \leq ka\) where \(a \in E_+\) and \(k \in (0,1)\) then \(a = 0\).

**LEMMA 1.7 \cite{14}:** Let \(P\) and \(Q\) are weakly compatible self-maps on a set \(Y\). If \(P\) and \(Q\) have a unique point of coincidence \(c = Pc = Qc\), then \(c\) is the unique common fixed point of \(P\) and \(Q\).

**2. MAIN RESULTS:** In this section, we prove some fixed point theorems for three mappings in vector \(b\)-metric space. Kir and Kiziltunc\cite{12} have investigated common fixed point theorems for weakly compatible pairs for \(b\)-metric space, whereas these results on vector metric spaces have been investigated by Rad and Altun\cite{15}.

**THEOREM 2.1:** Let \(X\) be \(E\)-b-metric space with \(E\)-Archimedean. Suppose the mappings \(P, Q, R : X \rightarrow X\) satisfy the following conditions:

(i) for all \(x, y \in X\), \(d(Px, Qy) \leq tM_{x,y}(P, Q, R)\) where \(t < \frac{1}{s(s+1)}\) and

\[M_{x,y}(P, Q, R) \in \{d(Rx, Ry), d(Px, Rx), d(Qy, Ry), d(Px, Ry), d(Qy, Rx)\}\]

(ii) \(P(X) \cup Q(X) \subseteq R(X)\)

(iii) \(R(X)\) is an \(E\)-complete subspace of \(X\).

Then \(\{P, R\}\) and \(\{Q, R\}\) have a unique point of coincidence in \(X\). Moreover, if \(\{P, R\}\) and \(\{Q, R\}\) are weakly compatible, then \(P, Q\) and \(R\) have a unique fixed point in \(X\).

**PROOF:** Let \(x_0\) be arbitrary point of \(X\). Since \(P(X) \subseteq R(X)\) there exists \(x_1 \in X\) such that \(P(x_0) = Rx_1 = y_1\).

Since \(Q(X) \subseteq R(X)\) there exists \(x_2 \in X\) such that \(Q(x_1) = Rx_2 = y_2\).

Continue in this manner, then there exists \(x_{2n+1} \in X\) such that \(P(x_{2n}) = Rx_{2n+1} = y_{2n+1}\), there exists \(x_{2n+2} \in X\) such that \(Q(x_{2n+1}) = Rx_{2n+2} = y_{2n+2}\), for \(n = 0, 1, 2, 3, \ldots\).

Firstly, show that

\[d(y_{2n+1}, y_{2n+2}) \leq \beta d(y_{2n}, y_{2n+1})\]

for all \(n\) where \(\beta < 1\) \hspace{1cm} (3)

From (1), we have:

\[d(y_{2n+1}, y_{2n+2}) = d(Px_{2n}, Qx_{2n+1}) \leq tM_{x_{2n}, x_{2n+1}}(P, Q, R)\]

for \(n = 0, 1, 2, 3, \ldots\).
Since $M_{x_n x_{n+1}}(P, Q, R) \in \{d(Rx_{2n}, Rx_{2n+1}), d(Px_{2n}, Rx_{2n}), d(Qx_{2n+1}, Rx_{2n+1}), d(Px_{2n}, Rx_{2n+1}),$
\[d(Qx_{2n+1}, Rx_{2n})\} = \{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+2}), d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+1})\}$
\[d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+1})\}$
\[d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+1})\}$

If $M_{x_n x_{n+1}}(P, Q, R) = d(y_{2n}, y_{2n+1}),$ then clearly (3) holds.

If $M_{x_n x_{n+1}}(P, Q, R) = d(y_{2n+1}, y_{2n+2}),$ then according to lemma 1.6
\[d(y_{2n+1}, y_{2n+2}) = 0,
\]and clearly (3) holds.

Finally, suppose that $M_{x_n x_{n+1}}(P, Q, R) = d(y_{2n}, y_{2n+2}),$
Then, we have
\[d(y_{2n+1}, y_{2n+2}) \leq td(y_{2n}, y_{2n+1}) \leq ts[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n})]
\]
\[d(y_{2n}, y_{2n+1}), \text{ where } \beta = \left(\frac{ts}{1-ts}\right)
\]
Thus $d(y_{n}, y_{n+1}) \leq \beta^n d(y_0, y_1), \text{ where } \beta \in \left(1, \frac{ts}{1-ts}\right)$

Therefore for all n and p,
\[d(y_{n}, y_{n+p}) \leq s d(y_{n}, y_{n+1}) + s^2 d(y_{n+1}, y_{n+2}) + s^3 d(y_{n+2}, y_{n+3}) + \ldots + s^p d(y_{n+p-1}, y_{n+p})
\]
\[\leq s \beta^n d(y_0, y_1) + s^2 \beta^{n+1} d(y_0, y_1) + \ldots + s^p \beta^{n+p-1} d(y_0, y_1)
\]
\[= s \beta^n \left(\frac{1-(s \beta)^p}{1-s \beta}\right) d(y_0, y_1)
\]
\[\leq \left(\frac{s \beta^n}{1-s \beta}\right) d(y_0, y_1)
\]

Since E is Archimedean, then $(y_n)$ is E-Cauchy sequence. Suppose that R(X) is E-complete, there exists a $p \in R(X)$ such that
\[Rx_{2n} = y_{2n} \xrightarrow{\delta} p \text{ and } Rx_{2n+1} = y_{2n+1} \xrightarrow{\delta} p
\]

Hence there exists a sequence $(c_n)$ in E such that $c_n \downarrow 0$ and $d(Rx_{2n}, p) \leq c_n, d(Rx_{2n+1}, p) \leq c_{n+1}.$ Since $p \in R(X),$ there exists $k \in X$ such that $Rk = p.$ Now we prove that $Qk = p
\]

For this, consider
\[d(p, Qk) \leq sd(p, Px_{2n}) + sd(Px_{2n}, Qk)
\]
\[\leq sc_{n+1} + stM_{x_{2n}, k}(P, Q, R)
\]
where \( M_{x_{2n},R}(P,Q,R) \in \{ d(Rx_{2n},R_k), d(Px_{2n},Rx_{2n}), d(Qk,Rk), d(Px_{2n},R_k), d(Qk,Rx_{2n}) \} \)
\[ = \{ d(y_{2n},p), d(y_{2n+1},y_{2n}), d(Qk,p), d(y_{2n+1},p), d(Qk,y_{2n}) \} \]
for all \( n \).

There are five possibilities:

Case 1: \( d(p,Qk) \leq sc_{n+1} + st \{ d(y_{2n+1},p) \leq sc_{n+1} + stc_n \leq (t+1)c_n \} \)

Case 2: \( d(p,Qk) \leq sc_{n+1} + st \{ d(y_{2n+1},y_{2n}) \leq sc_{n+1} + st \{ sd(y_{2n+1},p) + sd(p,y_{2n}) \} \}
\leq sc_{n+1} + st \{ sc_{n+1} + sc_n \} \leq s(2st+1)c_n \)

Case 3: \( d(p,Qk) \leq sc_{n+1} + std(p,Qk) \)
\[ (1-st)d(p,Qk) \leq sc_{n+1} \]
\[ d(p,Qk) \leq \left( \frac{s}{1-st} \right) c_{n+1} \]

Case 4: \( d(p,Qk) \leq sc_{n+1} + st \{ d(Qk,y_{2n}) \}
\leq sc_{n+1} + st \{ sd(Qk,p) + sd(p,y_{2n}) \}
\leq sc_{n+1} + st \{ sd(Qk,p) + sd(p,y_{2n}) \}
\leq sc_{n+1} + st \{ sc_{n+1} + sc_n \}
\leq s(t+1)c_n \)

Case 5: \( d(p,Qk) \leq sc_{n+1} + st \{ d(Qk,y_{2n}) \}
\leq sc_{n+1} + st \{ sd(Qk,p) + sd(p,y_{2n}) \}
\leq sc_{n+1} + st \{ sd(Qk,p) + sd(p,y_{2n}) \}
\leq sc_{n+1} + st \{ sc_{n+1} + sc_n \}
\leq s(t+1)c_n \)

Since the infimum of the sequences on the right hand side are zero, then \( d(p,Qk) = 0 \), that is Qk = p. Therefore Qk = Rk = p, i.e. p is a point of coincidence of mappings Q, R and k is a coincidence point of mappings Q and R.

Now we show that Pk = p, consider
\[ d(Pk,p) \leq sd(Pk,Qx_{2n+1}) + sd(Qx_{2n+1},p) \leq sc_{n+1} + stM_{x_{2n+1},2n+1}(P,Q,R) \]
where \( M_{x_{2n+1},2n+1}(P,Q,R) \in \{ d(Rk,Rx_{2n+1}), d(Pk,Rk), d(Qx_{2n+1},Rx_{2n+1}), d(Pk,Rx_{2n+1}), d(Qx_{2n+1},Rk) \} \)
\[ = \{ d(p,y_{2n+1}), d(Pk,p), d(y_{2n+2},y_{2n+1}), d(Pk,y_{2n+1}), d(Qx_{2n+1},p) \} \]
for all \( n \).

There are five possibilities:

Case 1: \( d(Pk,p) \leq sc_{n+1} + std(p,y_{2n+1}) \leq sc_{n+1} + stc_n \leq s(t+1)c_n \)

Case 2: \( d(Pk,p) \leq sc_{n+1} + std(Pk,p) \)
\[ (1-st)d(Pk,p) \leq sc_{n+1} \]
\[ d(Pk,p) \leq \left( \frac{s}{1-st} \right) c_{n+1} \]
Case 3: \(d(P_k, p) \leq sc_{n+1} + \text{std}(y_{2n+2}, y_{2n+1}) \leq sc_{n+1} + st[sd(y_{2n+2}, p) + sd(p, y_{2n+1},)]\)
\[d(P_k, p) \leq sc_{n+1} + st[sc_{n+2} + sc_{n+1}]\]
\[d(P_k, p) \leq sc_{n+1} + s^2tsc_{n+1} \leq s(st+1) c_{n+1}.
\]

Case 4: \(d(P_k, p) \leq sc_{n+1} + \text{std}(y_{2n+2}, y_{2n+1}) \leq sc_{n+1} + s^2td(P_k, p) + s^2tsc_{n+1}\)
\[d(P_k, p) \leq sc_{n+1} + st[sc_{n+2} + sc_{n+1}]\]
\[d(P_k, p) \leq n 1 \frac{1}{2} s(1 st) c_{n+1}.\]

Case 5: \(d(P_k, p) \leq sc_{n+1} + \text{std}(Qx_{2n+1}, p) \leq sc_{n+1} + stc_{n+1} \leq s(1 + t)c_{n+1}.
\]

Since the infimum of the sequences on the right hand side are zero, then \(d(P_k, p) = 0\), that is \(P_k = p\).

Therefore \(P_k = R_k = p\), i.e. \(p\) is a point of coincidence of mappings \(P, R\) and \(k\) is a coincidence point of mappings \(P\) and \(R\).

Now it remains to prove that \(p\) is a unique point of coincidence of pairs \(\{P,R\}\) and \(\{Q,R\}\).

Let \(p'\) be also a point of coincidence of these three mappings, then \(P_k' = Qk' = Rk' = p'\), for \(k' \in X\), we have,
\[d(p, p') = d(P_k, Qk') \leq tM_{k,k'}(P,Q,R)\]
\[d(P_k, p) \leq \left(\frac{s(1 + st)}{(1 - s^2t)}\right)c_{n+1}.
\]

If \(\{P,R\}\) and \(\{Q,R\}\) are weakly compatible, then \(p\) is a unique common fixed point of \(P,Q\) and \(R\).

**COROLLARY 2.2:** Let \(X\) be \(E\)-b-metric space with \(E\) Archimedean. Suppose the mappings \(P, R : X \to X\) satisfy the following conditions :

(i) for all \(x, y \in X\), \(d(Px, Py) \leq tM_{x,y}(P, R)\) \hspace{1cm} (4)

where \(t < \frac{1}{s(s+1)}\)

\[M_{x,y}(P, R) \in \{d(Rx, Ry), d(Px, Rx), d(Py, Ry), d(Px, Ry), d(Py, Rx)\}\] \hspace{1cm} (5)

(ii) \(P(X) \subseteq R(X)\)

(iii) \(R(X)\) is \(E\)-complete subspace of \(X\).

Then \(\{P, R\}\) have a unique point of coincidence in \(X\). Moreover, if \(\{P, R\}\) are weakly compatible, then they have a unique fixed point in \(X\).

**EXAMPLE 2.3:** Let \(E=R^2\) with coordinatewise ordering defined by \((x_1, y_1) \leq (x_2, y_2)\) if and only if \(x_1 \leq x_2\) and \(y_1 \leq y_2\), \(X = R\) and \(d(x, y) = (|x-y|^2, c|x-y|^2)\) with \(c > 0\).

Define the mappings \(Px = x^2 + 3, Rx = 2x^2\).
For all $x, y \in X$, we have
\[
d(Px, Py) = \frac{1}{2} d(Rx, Ry) \leq t M_{x,y}(P, R)
\]
with $M_{x,y}(P, R) = d(Rx, Ry)$ for $k \in \left[\frac{1}{2}, 1\right]$.

Moreover, $P(X) = [3, \infty) \subset [0, \infty) = R(X)$.

**THEOREM 2.4**: Let $X$ be $E$-b-metric space with $E$ Archimedean. Suppose the mappings $P, Q, R : X \to X$ satisfy the following conditions:

(i) for all $x, y \in X$, $d(Px, Qy) \leq t M_{x,y}(P, Q, R)$ \hspace{1cm} (6)

where $t < \frac{2}{s(s+2)}$ and

\[
M_{x,y}(P, Q, R) \in \left\{ \frac{1}{2} [d(Rx, Ry) + d(Px, Rx)], \frac{1}{2} [d(Rx, Ry) + d(Px, Ry)], \frac{1}{2} [d(Rx, Ry) + d(Qy, Rx)], \frac{1}{2} [d(Rx, Ry) + d(Qy, Ry)], \frac{1}{2} [d(Px, Rx) + d(Qy, Ry)], \frac{1}{2} [d(Px, Ry) + d(Qy, Rx)] \right\}
\]

(ii) $P(X) \cup Q(X) \subseteq R(X)$

(iii) $R(X)$ is an $E$-complete subspace of $X$.

Then $\{P, R\}$ and $\{Q, R\}$ have a unique common point of coincidence in $X$. Moreover, if $\{P, R\}$ and $\{Q, R\}$ are weakly compatible, then they have a unique fixed point in $X$.

**PROOF**: We define the sequence $\{x_n\}$ and $\{y_n\}$ as in proof of Theorem 2.1

Firstly, show that
\[
d(y_{2n+1}, y_{2n+2}) \leq \beta d(y_{2n}, y_{2n+1}) \text{ for all } n.
\]

(8)

From (6), we have:
\[
d(y_{2n+1}, y_{2n+2}) = d(Px_{2n}, Qx_{2n+1}) \leq t M_{x_{2n}, x_{2n+1}}(P, Q, R) \text{ for } n = 0, 1, 2, 3, \ldots
\]

Since
\[
M_{x_{2n}, x_{2n+1}}(P, Q, R) \in \left\{ \frac{1}{2} [d(Rx_{2n}, Rx_{2n+1}) + d(Px_{2n}, Rx_{2n})], \frac{1}{2} [d(Rx_{2n}, Rx_{2n+1}) + d(Px_{2n}, Rx_{2n+1})], \frac{1}{2} [d(Rx_{2n}, Rx_{2n+1}) + d(Qx_{2n+1}, Rx_{2n+1})], \frac{1}{2} [d(Px_{2n}, Rx_{2n}) + d(Qx_{2n+1}, Rx_{2n+1})], \frac{1}{2} [d(Px_{2n}, Rx_{2n+1}) + d(Qx_{2n+1}, Rx_{2n+1})] \right\}
\]
\[ \frac{1}{2} [d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n})], \frac{1}{2} [d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+1})], \frac{1}{2} [d(y_{2n}, y_{2n+1}) + d(y_{2n+2}, y_{2n})], \frac{1}{2} [d(y_{2n+1}, y_{2n+1}) + d(y_{2n+2}, y_{2n})] \]

\[ = \{ \frac{1}{2} [d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n})], \frac{1}{2} [d(y_{2n}, y_{2n+1}) + d(y_{2n+2}, y_{2n})], \frac{1}{2} [d(y_{2n+1}, y_{2n}) + d(y_{2n+2}, y_{2n+1})], \frac{1}{2} [d(y_{2n+1}, y_{2n}) + d(y_{2n+2}, y_{2n})] \} \]

If \( M_{X_{2n-2n+1}}(P,Q,R) = d(y_{2n}, y_{2n+1}) \) or \( \frac{1}{2} [d(y_{2n}, y_{2n+1})] \) then clearly (8) holds.

If \( M_{X_{2n-2n+1}}(P,Q,R) = \frac{1}{2} [d(y_{2n}, y_{2n+1}) + d(y_{2n+2}, y_{2n})] \)

Then \( d(y_{2n+1}, y_{2n+2}) \leq \frac{1}{2} [d(y_{2n}, y_{2n+1})] + \frac{1}{2} [d(y_{2n+2}, y_{2n})] \)

\[ \left( 1 - \frac{st}{2} \right) d(y_{2n+1}, y_{2n+2}) \leq (1 + s) \frac{t}{2} [d(y_{2n}, y_{2n+1})] \]

\[ d(y_{2n+1}, y_{2n+2}) \leq \frac{t}{2} \left( \frac{1 + s}{1 - \frac{st}{2}} \right) [d(y_{2n}, y_{2n+1})] \leq \beta' [d(y_{2n}, y_{2n+1})], \quad \text{where} \quad \beta' = \frac{t}{2} \left( \frac{1 + s}{1 - \frac{st}{2}} \right) \]

If \( M_{X_{2n-2n+1}}(P,Q,R) = \frac{1}{2} [d(y_{2n}, y_{2n+1})+d(y_{2n+2}, y_{2n+1})] \)

Then \( d(y_{2n+1}, y_{2n+2}) \leq \frac{1}{2} [d(y_{2n}, y_{2n+1})] + \frac{1}{2} [d(y_{2n+2}, y_{2n+1})] \)

\[ \left( 1 - \frac{t}{2} \right) d(y_{2n+1}, y_{2n+2}) \leq \frac{1}{2} [d(y_{2n}, y_{2n+1})] \]

\[ d(y_{2n+1}, y_{2n+2}) \leq \left( \frac{t}{2} \right) \left( \frac{1}{1 - \frac{t}{2}} \right) [d(y_{2n}, y_{2n+1})] \leq \beta'' [d(y_{2n}, y_{2n+1})], \quad \text{where} \quad \beta'' = \left( \frac{t}{2} \right) \left( \frac{1}{1 - \frac{t}{2}} \right) \]

If \( M_{X_{2n-2n+1}}(P,Q,R) = \frac{1}{2} [d(y_{2n}, y_{2n+2})] \)

Then \( d(y_{2n+1}, y_{2n+2}) \leq \frac{1}{2} [sd(y_{2n}, y_{2n+1}) + sd(y_{2n+1}, y_{2n+2})] \)
\[d(y_{2n+1}, y_{2n+2}) \leq \left(\frac{st}{2}\right) \frac{[d(y_{2n}, y_{2n+1})]}{1 - \frac{st}{2}} \leq \beta''' [d(y_{2n}, y_{2n+1})], \quad \text{where } \beta''' = \left(\frac{st}{2}\right) \frac{[d(y_{2n}, y_{2n+1})]}{1 - \frac{st}{2}}.\]

Therefore, \(d(y_{n}, y_{n+1}) \leq (\beta''')^n d(y_0, y_1) \quad \text{(9)}\)

By using (9), for all \(n\) and \(p\), we have

\[d(y_{n}, y_{n+p}) \leq s d(y_{n}, y_{n+1}) + s^2 d(y_{n+1}, y_{n+2}) + \ldots + s^p d(y_{n+p-1}, y_{n+p}) \leq s (\beta''')^n d(y_0, y_1) \quad \text{(10)}\]

Since \(E\) is Archimedean, then \((y_n)\) is \(E\)-Cauchy sequence. Suppose that \(R(X)\) is \(E\)-complete, there exists a \(q \in R(X)\) such that

\[R_{x_n} = y_n \xrightarrow{d.E} q\text{ and } R_{x_{n+1}} = y_{n+1} \xrightarrow{d.E} q\]

Hence there exists a sequence \((c_n)\) in \(E\) such that \(c_n \downarrow 0\) and \(d(R_{x_n}, q) \leq c_n\), \(d(R_{x_{n+1}}, q) \leq c_{n+1}\). Since \(q \in R(X)\), there exists \(k \in X\) such that \(R_k = q\). Now we prove that \(Q_k = q\) for this, consider

\[d(q, Q_k) \leq s d(q, P_{x_n}) + s d(P_{x_n}, Q_k) \leq s c_{n+1} + s t M_{x_{2n+k}}(P, Q, R)\]

where \(M_{x_{2n+k}}(P, Q, R) \in \{ \frac{1}{2} [d(R_{x_{2n}}, R_k) + d(P_{x_{2n}}, R_{x_{2n}})], \frac{1}{2} [d(R_{x_{2n}}, R_k) + d(P_{x_{2n}}, R_{x_{2n}})], \frac{1}{2} [d(R_{x_{2n}}, R_k) + d(Q_{x_{2n}}, R_{x_{2n}})], \frac{1}{2} [d(P_{x_{2n}}, R_{x_{2n}}) + d(Q_{x_{2n}}, R_{x_{2n}})], \frac{1}{2} [d(P_{x_{2n}}, R_k) + d(Q_{x_{2n}}, R_{x_{2n}})]\}

\[= \{ \frac{1}{2} [d(y_{2n}, q) + d(y_{2n+1}, y_{2n})], \frac{1}{2} [d(y_{2n}, q) + d(y_{2n+1}, q)], \frac{1}{2} [d(y_{2n}, q) + d(Q_k, y_{2n})], \frac{1}{2} [d(y_{2n+1}, q) + d(Q_k, q)], \frac{1}{2} [d(y_{2n+1}, q) + d(Q_k, y_{2n})]\}

There are six possibilities:

Case 1: \(d(q, Q_k) \leq s c_{n+1} + \frac{st}{2} [d(y_{2n}, q) + d(y_{2n+1}, y_{2n})]\)

\[\leq s c_{n+1} + \frac{st}{2} c_n + \frac{st}{2} [s d(y_{2n+1}, q) + s d(q, y_{2n})]\]

\[\leq s c_{n+1} + \frac{st}{2} c_n + \frac{s^2 t}{2} c_{n+1} + \frac{s^2 t}{2} s c_n\]

\[\leq s (1 + \frac{t}{2} + st)c_n\]
Case 2: \( d(q, Q_k) \leq sc_{n+1} + \frac{st}{2} [d(y_{2n}, q) + d(y_{2n+1}, q)] \)

\[ \leq sc_{n+1} + \frac{st}{2} c_{n+1} \]

\[ \leq s(t+1) c_n. \]

Case 3: \( d(q, Q_k) \leq sc_{n+1} + \frac{st}{2} [d(y_{2n}, q) + d(Q_k, y_{2n})] \)

\[ \leq sc_{n+1} + \frac{st}{2} c_n + \frac{st}{2} c_{n+1} \leq s(t+1) c_n. \]

\( \left(1 - \frac{s^2 t}{2}\right) d(q, Q_k) \leq sc_{n+1} + \frac{st}{2} c_n + \frac{s^2 t}{2} c_n \)

\[ d(q, Q_k) \leq s \left(1 + \frac{t + \frac{st}{2}}{2} \right) c_n \]

Case 4: \( d(q, Q_k) \leq sc_{n+1} + \frac{st}{2} [d(y_{2n}, q) + d(Q_k, q)] \)

\[ \left(1 - \frac{st}{2}\right) d(q, Q_k) \leq sc_{n+1} + \frac{st}{2} c_n \]

\[ d(q, Q_k) \leq s \left(1 + \frac{t + \frac{st}{2}}{2} \right) c_n \]

Case 5: \( d(q, Q_k) \leq sc_{n+1} + \frac{st}{2} [d(y_{2n+1}, y_{2n}) + d(Q_k, q)] \)

\[ \left(1 - \frac{st}{2}\right) d(q, Q_k) \leq sc_{n+1} + \frac{s^2 t}{2} c_{n+1} + \frac{s^2 t}{2} c_n \]

\[ d(q, Q_k) \leq s \left(1 + \frac{t + \frac{st}{2}}{2} \right) c_n \]

Case 6: \( d(q, Q_k) \leq sc_{n+1} + \frac{st}{2} [d(y_{2n+1}, q) + d(Q_k, y_{2n})] \)

\[ \leq sc_{n+1} + \frac{st}{2} c_{n+1} + \frac{st}{2} [sd(Q_k, q) + sd(q, y_{2n})] \]
\[
\left(1 - \frac{s^2}{2}\right) d(q, Qk) \leq sc_{n+1} + \frac{st}{2} c_{n+1} + \frac{s^2}{2} c_n
\]

\[
d(q, Qk) \leq s \left( \frac{1 + \frac{t}{2} + \frac{st}{2}}{1 - \frac{s^2}{2}} \right) c_{n+1}
\]

Since the infimum of the sequences on the right hand side are zero, therefore \(d(q, Qk) = 0\), that is \(Qk = q\). Therefore \(Qk = Rk = q\) i.e. \(q\) is a point of coincidence of mappings \(Q\), \(R\) and \(k\) is a coincidence point of mappings \(Q\) and \(R\).

Now we show that \(Pk = q\).

Consider, \(d(Pk, q) \leq sd(Pk, Qx_{2n+1}) + sd(Qx_{2n+1}, q) \leq sc_{n+1} + stM_{x_k2n+1}(P, Q, R)\)

where \(M_{x_k2n+1}(P, Q, R) \in \left\{ \frac{1}{2} [d(Rk, Qx_{2n+1}) + d(Pk, Rk)], \frac{1}{2} [d(Rk, Rx_{2n+1}) + d(Pk, Rx_{2n+1})], \frac{1}{2} [d(Pk, Rk) + d(Qx_{2n+1}, Rx_{2n+1})], \frac{1}{2} [d(Rk, Rx_{2n+1}) + d(Qx_{2n+1}, Rk)] \right\}\n
\[
\frac{1}{2} [d(Pk, Rk) + d(Qx_{2n+1}, Rk)], \frac{1}{2} [d(Pk, Rx_{2n+1}) + d(Qx_{2n+1}, Rk)]
\]

\[
= \left\{ \frac{1}{2} [d(q, y_{2n+1}) + d(Pk, q)], \frac{1}{2} [d(q, y_{2n+1}) + d(Pk, y_{2n+1})], \frac{1}{2} [d(q, y_{2n+1}) + d(y_{2n+2}, q)], \frac{1}{2} [d(q, y_{2n+2}) + d(y_{2n+2}, y_{2n+1})], \frac{1}{2} [d(Pk, q) + d(y_{2n+2}, y_{2n+1})], \frac{1}{2} [d(Pk, y_{2n+1}) + d(y_{2n+2}, q)] \right\}
\]

There are six possibilities:

Case 1: \(d(Pk, q) \leq sc_{n+1} + \frac{st}{2} [d(q, y_{2n+1}) + d(Pk, q)]\)

\[
\left(1 - \frac{st}{2}\right) d(Pk, q) \leq sc_{n+1} + \frac{st}{2} c_{n+1}
\]

\[
d(Pk, q) \leq s \left( \frac{1 + \frac{t}{2} + \frac{st}{2}}{\left(1 - \frac{st}{2}\right)} \right) c_{n+1}
\]

Case 2: \(d(Pk, q) \leq sc_{n+1} + \frac{st}{2} [d(q, y_{2n+1}) + d(Pk, y_{2n+1})]\)

\[
d(Pk, q) \leq sc_{n+1} + \frac{st}{2} c_{n+1} + \frac{st}{2} \left[ sd(Pk, q) + sd(q, y_{2n+1}) \right]
\]
\[
\left(1 - \frac{s^2t}{2}\right) d(P_k, q) \leq sc_{n+1} + \frac{st}{2} c_{n+1} + \frac{s^2t}{2} c_{n+1}
\]
\[
d(P_k, q) \leq s \left( \frac{1 + \frac{t}{2} + \frac{st}{2}}{1 - \frac{s^2t}{2}} \right) c_n
\]

Case 3: \(d(P_k, q) \leq sc_{n+1} + \frac{st}{2} [d(q, y_{2n+1}) + d(y_{2n+2}, q)] \leq sc_{n+1} + \frac{st}{2} c_{n+1} + \frac{st}{2} c_{n+1}
\]
\[
d(P_k, q) \leq s(1 + t)c_{n+1}
\]

Case 4: \(d(P_k, q) \leq sc_{n+1} + \frac{st}{2} [d(q, y_{2n+1}) + d(y_{2n+2}, y_{2n+1})]
\]
\[
\leq sc_{n+1} + \frac{st}{2} c_{n+1} + \frac{s^2t}{2} c_{n+1} + \frac{s^2t}{2} c_{n+1}
\]
\[
\leq s(1 + st + \frac{t}{2})c_{n+1}
\]

Case 5: \(d(P_k, q) \leq sc_{n+1} + \frac{st}{2} [d(P_k, q) + d(y_{2n+2}, y_{2n+1})]
\]
\[
\leq sc_{n+1} + \frac{st}{2} [(P_k, q)] + \frac{st}{2} [sd(y_{2n+2}, q) + sd(q, y_{2n+1})]
\]
\[
\left(1 - \frac{st}{2}\right) d(P_k, q) \leq sc_{n+1} + \frac{s^2t}{2} c_{n+1} + \frac{s^2t}{2} c_{n+1}
\]
\[
d(P_k, q) \leq s \left( \frac{1 + \frac{t}{2} + \frac{st}{2}}{1 - \frac{s^2t}{2}} \right) c_{n+1}
\]

Case 6: \(d(P_k, q) \leq sc_{n+1} + \frac{st}{2} [d(P_k, y_{2n+1}) + d(y_{2n+2}, q)]
\]
\[
d(P_k, q) \leq sc_{n+1} + \frac{st}{2} [sd(P_k, q) + sd(q, y_{2n+1})] + \frac{st}{2} c_{n+1}
\]
\[
\left(1 - \frac{s^2t}{2}\right) d(P_k, q) \leq s \left( \frac{1 + \frac{t}{2} + \frac{st}{2}}{1 - \frac{s^2t}{2}} \right) c_{n+1}
\]
Since the infimum of the sequences on the right hand side are zero, therefore \( d(P_k, q) = 0 \), that is \( P_k = q \). Therefore \( P_k = R_k = q \), i.e. \( n \) is a point of coincidence of mappings \( P \) and \( R \). Thus \( k \) is a coincidence point of mappings \( P \) and \( R \).

Now it remains to prove that \( q \) is a unique point of coincidence of pairs \( \{ P, R \} \) and \( \{ Q, R \} \).

Let \( q' \) be also a point of coincidence of these three mappings, then \( P_k' = Q_k' = T_k' = q' \), for \( k' \in X \), we have,

\[
d(q, q') = d(P_k, Q_k') \leq tM_{k,k'}(P, Q, R)
\]

where \( M_{k,k'}(P, Q, R) \in \{ \frac{1}{2} [d(R_k, R_k') + d(P_k, R_k)], \frac{1}{2} [d(R_k, R_k') + d(P_k, R_k')], \frac{1}{2} [d(R_k, R_k') + d(Q_k', R_k')], \frac{1}{2} [d(R_k, R_k') + d(Q_k', R_k')], \frac{1}{2} [d(P_k, R_k) + d(Q_k, R_k')], \frac{1}{2} [d(P_k, R_k') + d(Q_k', R_k')] \} = \{ 0, d(q, q') \}

Hence \( d(q, q') = 0 \) i.e. \( q = q' \)

If \( \{ P, R \} \) and \( \{ Q, R \} \) are weakly compatible, then \( q \) is a unique common fixed point of \( P, Q \) and \( R \).

3. RESULTS AND DISCUSSION

In 2016, Rad and Altun\(^{15}\) proved some common fixed point results for three mappings on vector metric spaces. They proved the following results:

**THEOREM 3.1:** Let \( X \) be a vector metric space with \( E \)-Archimedean. Suppose the mappings \( f, g, T : X \rightarrow X \) satisfy the following conditions :

(i) for all \( x, y \in X \), \( d(fx, gy) \leq ku_{x,y}(f, g, T) \) \hspace{1cm} (10)

where \( k \in (0, 1) \) is a constant and

\[
u_{x,y}(f,g,T) \in \{ d(Tx, Ty), d(fx, Tx), d(gy, Ty), \frac{1}{2} [d(fx, Ty) + d(gy, Tx)] \} \hspace{1cm} (11)
\]

(ii) \( f(X) \cup g(X) \subseteq T(X) \)

(iii) one of \( f(X) \), \( g(X) \) or \( T(X) \) is a \( E \)-complete subspace of \( X \).

Then \( \{ f, T \} \) and \( \{ g, T \} \) have a unique point of coincidence in \( X \). Moreover, if \( \{ f, T \} \) and \( \{ g, T \} \) are weakly compatible, then \( f, g \) and \( T \) have a unique common fixed point in \( X \) where \( k \in (0, 1] \).

\[
u_{x,y}(f, g) \in \{ d(fx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx) \} \hspace{1cm} (13)
\]

(ii) \( f(X) \subseteq T(X) \)

(iii) one of \( f(X) \) or \( T(X) \) is \( E \)-complete subspace of \( X \).
Then \( \{f, T\} \) have a unique point of coincidence in \( X \). Moreover, if \( \{f, T\} \) are weakly compatible, then \( f \) and \( T \) have a unique common fixed point in \( X \).

In 2017, Latpate\(^1\) proved the results for three mappings on complete metric spaces. He proved the following result:

Let \((X, d)\) be a complete Metric space and Let \( A \) be a nonempty closed subset of \( X \).

Let \( P, Q : A \rightarrow A \) be such that

\[
d(P_x, Q_y) \leq \frac{1}{2} [d(R_x, Q_y) + d(R_y, P_x) + d(S_x, R_y)] - \psi[d(R_x, Q_y) + d(R_y, P_x)] \tag{14}\]

For any \((x, y) \in X \times X\), where a function \( \psi : [0, \infty)^2 \rightarrow [0, \infty) \) is continuous and \( \psi(x, y) = 0 \) iff \( x = y = 0 \) and \( R : A \rightarrow X \) which satisfies the following condition.

(i) \( PA \subseteq RA \) and \( QA \subseteq RA \)

(ii) The pair of mappings \((P, R)\) and \((Q, R)\) are weakly compatible.

(iii) \( R(A) \) is closed subset of \( X \).

Then \( P, R \) and \( Q \) have unique common fixed point.

Motivated by their results, we have proved similar results for three mappings on E-b-metric spaces.

Further, these results can be investigated for four and six mappings on E-b-metric space.

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**REFERENCES**