**K-Contra Harmonic Mean Labeling of Some Graphs**

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**ABSTRACT**

Let $G$ be a $(p, q)$ graph. A function $f$ is called a k-contra harmonic mean labelling of a graph $G$ if $f: V(G) \rightarrow \{k, k+1, k+2, ..., k + q\}$ in such a way that the function defined as,

$$f^* : E(G) \rightarrow \{k, k + 1, k + 2, ..., k + q - 1\}$$

is an edge labeling. The graph which admits $k$-contra harmonic mean labelling is called $k$-Contra harmonic mean graph.

**KEYWORDS**: $k$-Contra Harmonic mean labeling, $K$-Contra Harmonic mean graphs, Path, Cycle, Comb, etc.

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1. INTRODUCTION

By a graph $G = (V(G), E(G))$ with $p$ vertices and $q$ edges we mean a simple, connected and undirected graph. In this paper a brief summary of definitions and other information is given in order to maintain compactness. The term not defined here are used in the sense of Harary $^2$.

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. A useful survey on graph labeling by J.A. Gallian (2016) can be found in $^1$. If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling).

All graphs in this paper are simple, finite, undirected. Let $G$ be a graph with $p$ vertices and $q$ edges. For a detail survey of graph labeling we refer to Gallian $^1$. For all other standard terminology and notation we follow Harary $^2$. S. Somasundaram and R. Ponraj introduced mean labeling for some standard graphs in 2013. S.S. Sandhya and S. Somasundaram introduced Harmonic mean labeling of graph. S. S. Sandhya, S. Somasundaram and J. Rajeshni Golda introduced Contra Harmonic mean labeling of graphs $^9$.

We have introduced K- Contra Harmonic mean labeling. In this paper we investigate the $k$-Contra Harmonic mean labeling behaviour of some special graphs. The following definitions are useful for our present study.

**Definition 1.1** Let $G$ be a $(p, q)$ graph. A function $f$ is called a $k$-contra harmonic mean labelling of a graph $G$ if $\{ f(v) : v \in V(G) \} = \{ k, k+1, k+2, \ldots, k+q \}$ in such a way that the function $f^*$ defined as

$$f^*(e = uv) = \left[ \frac{\sum_{i=1}^{k+q} f^i(e)}{\sum_{i=1}^{k+q} f^i(u)} \right]$$

with distinct edge labels. The graph which admits $k$-contra harmonic mean labeling is called $k$-contra harmonic mean graph.

**Definition 1.2** The union of two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ is a graph $G=G_1 \cup G_2$ with vertex set $V=V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.

**Definition 1.3** The corona of two graphs $G_1$ and $G_2$ is the graph $G = G_1 \circ G_2$ formed by taking one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$ where the $i^{th}$ vertex of $G_1$ is adjacent to every vertex in the $i^{th}$ copy of $G_2$.

**Definition 1.4** A Triangular ladder $\mathcal{TL}_{n,n} \geq 2$ is a graph obtained from a ladder $L_n$ by adding the edges $e_1, e_2, \ldots, e_n$ for $1 \leq i \leq n-1$ where $u_i$ and $v_i$ for $1 \leq i \leq n$ are the vertices of $L_n$. Such that $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ are two paths of length $n$ in $L_n$.

**Definition 1.5** An $(m, n)$ kite graph consists of cycle of length $m$ with $n$ edges path attached to one vertex of a cycle.
Definition 1.6 Comb is a graph obtained by joining a single pendant edge to each vertex of a path.

2. MAIN RESULTS

Theorem 2.1 The path $P_n$ is a k-contra harmonic mean graph for all k and $n \geq 2$.

Proof: Let $V(P_n) = \{v_i \mid 1 \leq i \leq n\}$ and $E(P_n) = \{e = v_i v_{i+1} \mid 1 \leq i \leq n-1\}$.

Define a function $f: V(G) \rightarrow \{k, k+1, k+2, \ldots, k+q\}$ by

$$f(v_i) = k + i - 1, \quad 1 \leq i \leq n$$

Then the induced edge labels are

$$f^e(u_l) = k + l - 1, \quad 1 \leq l \leq n - 1$$

The above defined function $f$ provides k- contra harmonic mean labeling of the graph. Hence $P_n$ is a k- contra harmonic mean graph.

Example 2.2

![500-harmonic mean labeling of $P_{10}$](image)

Theorem 2.3 The cycle graph $C_n$ is a k-contra harmonic mean graph.

Proof: Let $u_1, u_2, \ldots, u_n, u_1$ be the given cycle of length n.

Define a function $f: V(G) \rightarrow \{k, k+1, k+2, \ldots, k+q\}$ by

$$f(u_l) = k + l - 1, \text{ for } 1 \leq l \leq n-1,$$

$$f(u_l) = k + q, \text{ for } l = n.$$ 

Then the induced edge labels are

$$f^e(u_l u_{l+1}) = k + l - 1, \text{ for } 1 \leq l \leq n-2$$

$$f^e(u_l u_{l+2}) = k + q - 1, \text{ for } l = n-1$$

$$f^e(u_1 u_2) = k + q - 2, \text{ for } l = n$$

The above defined function $f$ provides k- contra harmonic mean labeling of the graph. Hence $C_n$ is a k- contra harmonic mean graph.
**Example 2.3**

\[\text{50-contra harmonic mean labeling of } C_5\]

**Theorem 2.4** The Triangular ladder \( T_{L_n} \) is \( k \)- contra harmonic mean graph for all \( k \) and \( n \geq 2 \).

**Proof:** Let \( V(T_{L_n}) = \{ u_i, v_i \mid 1 \leq i \leq n \} \) and

\[ E(T_{L_n}) = \{ u_iu_{i+1}, v_iv_{i+2}, u_iv_{i+2} \mid 1 \leq i \leq n-1 \} \cup \{ u_iv_i \mid 1 \leq i \leq n \}. \]

First we label the vertices as follows

Define a function \( f : V(G) \to \{ k, k+1, k+2, \ldots, k+q \} \) by

\[ f(u_i) = k + 4(t-3), \text{for } 1 \leq i \leq n \]
\[ f(v_i) = k \]
\[ f(v_i) = k + 4(t-5), \text{for } 2 \leq i \leq n \]

Then the induced edge labels are

\[ f'(u_iv_{i+2}) = k + 4(t-1), \text{for } 1 \leq i \leq n-1 \]
\[ f'(v_iv_{i+2}) = k + 4(t-3), \text{for } 1 \leq i \leq n-1 \]
\[ f'(u_iv_i) = k + 4(t-4), \text{for } 1 \leq i \leq n \]
\[ f'(u_iv_{i+2}) = k + 4(t-2), \text{for } 1 \leq i \leq n-1 \]

The above defined function \( f \) provides \( k \)- contra harmonic mean labeling of the graph.

Hence \( T_{L_n} \) is a \( k \)- contra harmonic mean graph.

**Example: 2.4**

\[700 \text{- Contra harmonic mean labeling of } T_{L_5}\]
Theorem 2.5A graph obtained by attaching a triangle at each pendent vertex of a comb is k- Contra harmonic mean graph for all k.

Proof: Let G be a graph obtained by attaching a triangle $K_2$ at each pendent vertex of $P_n \otimes K_1$. Let $u_i,v_i$ be the vertices of the comb $P_n \otimes K_1$ in which $v_i$ is joined with the vertex $u_i$ of $P_n$. Let $x_i,y_i,z_i$ be the vertices of $t^{th}$ copy of $K_3$. Identify $z_i$ with $v_i$. 1 ≤ t ≤ n.

The resultant graph is G whose edge set is

$E = \{ u_i u_{i+1} \mid 1 \leq i \leq n-1 \} \cup \{ u_i v_i, v_i x_i, v_i y_i, v_i z_i \mid 1 \leq i \leq n \}$.

Define a function $f : V(G) \rightarrow \{ k, k+1, k+2, \ldots, k + q \}$ by

\[
\begin{align*}
    f(u_i) &= k + 5i - 3, \text{ for } 1 \leq i \leq n \\
    f(v_i) &= k + 5i - 2, \text{ for } 1 \leq i \leq n \\
    f(x_i) &= k + 5i - 5, \text{ for } 1 \leq i \leq n \\
    f(y_i) &= k + 5i - 4, \text{ for } 1 \leq i \leq n \\
    f(z_i) &= k + 5i - 1, \text{ for } 1 \leq i \leq n - 1
\end{align*}
\]

Then the induced edge labels are

\[
\begin{align*}
    f^e(u_i u_{i+1}) &= k + 5i - 1, \text{ for } 1 \leq i \leq n - 1 \\
    f^e(u_i v_i) &= k + 5i - 2, \text{ for } 1 \leq i \leq n \\
    f^e(v_i x_i) &= k + 5i - 4, \text{ for } 1 \leq i \leq n \\
    f^e(v_i y_i) &= k + 5i - 3, \text{ for } 1 \leq i \leq n \\
    f^e(v_i z_i) &= k + 5i - 5, \text{ for } 1 \leq i \leq n
\end{align*}
\]

The above defined function $f$ provides k-contra harmonic mean labeling of the graph. Hence the graph G is k- contra harmonic mean graph.

Example: 2.6

\[
\begin{align*}
400 - \text{Contra harmonic mean labeling of G}
\end{align*}
\]

Theorem 2.7 $P_n \otimes K_1$ is k- contra harmonic mean labelling
Proof: Let \( v_1, v_2, \ldots, v_n \) be the path \( P_n \). Let \( v_i \) be the vertices which is joined to the vertex \( v_i, 1 \leq t \leq n \) of the path \( P_n \). The resultant graph is \( P_n \odot K_2 \).

Let \( G = P_n \odot K_2 \). Define a function \( f : V(G) \to \{k, k+1, k+2, \ldots, k+q\} \) by

\[
\begin{align*}
  f(v_t) &= k + 2t - 2 & \text{for } 1 \leq t \leq n \\
  f(v_1) &= k+2t-1 & \text{for } 1 \leq t \leq n 
\end{align*}
\]

Then the distinct edge labels are

\[
\begin{align*}
  f^{e}(v_1v_{t+1}) &= k + 2t - 1 & \text{for } 1 \leq t \leq n - 1 \\
  f^{e}(v_tv_{t+2}) &= k + 2t - 2 & \text{for } 1 \leq t \leq n 
\end{align*}
\]

The above defined function \( f \) provides \( k \)-contra harmonic mean labelling of the graph. Hence \( P_n \odot K_2 \) is \( k \)-contra harmonic mean labelling.

Example 2.8

![Diagram of 50-contra harmonic mean labelling of \( P_n \odot K_2 \).](image)

Theorem 2.9A Triangular snake \( T_n (n \geq 2) \) is \( k \)-contra harmonic mean graph \( \forall k \geq 2 \).

Proof: Let \( V(T_n) = \{u_t | 1 \leq t \leq n\} \cup \{v_t | 1 \leq t \leq n - 1\} \) and

\[ E(T_n) = \{u_t u_{t+2}, v_t v_{t+2} \mid 1 \leq t \leq n - 1\}. \]

First we label the vertices as follows.

Define a function \( f : V(T_n) \to \{k, k+1, k+2, \ldots, k+q\} \) by

\[
\begin{align*}
  f(u_t) &= k + 3t - 3 & \text{for } 1 \leq t \leq n \\
  f(v_t) &= k + 1 \\
  f(v_1) &= k + 3t - 2 & \text{for } 2 \leq t \leq n - 1 
\end{align*}
\]

Then the induced edge labels are

\[
\begin{align*}
  f^{e}(u_tu_{t+2}) &= k + 1 \\
  f^{e}(u_tu_{t+2}) &= k + 3t + 1 & \text{for } 2 \leq t \leq n - 1 \\
  f^{e}(v_tv_{t+2}) &= k + 3t - 3 & \text{for } 1 \leq t \leq n - 1 \\
  f^{e}(u_1v_1) &= k + 3t - 1 & \text{for } 2 \leq t \leq n - 1 \\
  f^{e}(u_2v_2) &= k + 2
\end{align*}
\]
The above defined function f provides k-contra harmonic mean labeling of the graph. Hence $T_n$ is a k– Contra harmonic mean graph.

**Example 2.10**

![100– Contra harmonic mean graph of $T_6$]

**Theorem 2.11** A $(m,n)$ kite graph $G$ is a k-contra harmonic mean graph.

**Proof:** Let $u_1, u_2, ..., u_m$ be the given cycle of length $m$ and $v_1, v_2, ..., v_n$ be the given path of length $n$.

Define a function $f : V(G) \rightarrow \{k, k+1, k+2, ..., k+q\}$ by

\[
\begin{align*}
  f(u_i) &= k + i - 1, \text{ for } 1 \leq i \leq m, \\
  f(v_i) &= k + i + 5, \text{ for } 1 \leq i \leq n.
\end{align*}
\]

Then the induced edge labels are

\[
\begin{align*}
  f^*(u_{i+1}) &= k + i - 1, \text{ for } 1 \leq i \leq n - 2, \\
  f^*(u_mu_m^{-1}) &= k + m - 1, \\
  f^*(u_1u_n) &= k + 3.
\end{align*}
\]

and the edge labels of the path are $\{k + m + 1, k + m + 2, ..., k + m + n - 1\}$. The above defined function $f$ provides k-contra harmonic mean labeling of the graph.

Hence the $(m,n)$ kite graph is a k-contra harmonic mean graph.

**Example 2.12**

![50 -contra harmonic mean labeling of (5,6) kite graph]
Theorem 2.13 Let $P_n$ be the path and $G$ be the graph obtained from $P_n$ by attaching $G_3$ in both the end edges of $P_n$. Then $G$ is a $k$-contra harmonic mean graph.

**Proof:** Let $P_n$ be the path $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$. Define a function $f : V(G) \rightarrow \{k, k + 1, k + 2, \ldots, k + q\}$ by

\[
\begin{align*}
 f(u_i) &= k + l, \text{ for } 1 \leq i \leq u_i, \\
 f(v_i) &= k; f(v_2) = k + q.
\end{align*}
\]

Then the induced edge labels are

\[
\begin{align*}
 f^e(u_1u_2) &= k + l + 1, \text{ for } 1 \leq i \leq n - 1 \\
 f^e(u_2v_2) &= k \\
 f^e(u_2v_3) &= k + 1 \\
 f^e(u_{n-2}v_2) &= k + n + 1 \\
 f^e(u_nv_2) &= k + n + 2
\end{align*}
\]

The above defined function $f$ provides $k$-contra harmonic mean labelling of the graph. Hence $G$ is a $k$-contra harmonic mean graph.

**Example 2.14:** A $k$-contra harmonic mean labelling of $G$ obtained from $P_n$ is

![200-contra harmonic mean labelling of G](image)

3. **CONCLUSION**

The Study of labelled graph is important due to its diversified applications. It is very interesting to investigate graphs which admit $k$-Contra Harmonic Mean Labelling. In this paper, we proved that Path, Triangular Ladder $TL_n$, a graph obtained by attaching a triangle at each pendant vertex of a comb, Comb, Triangular Snake, $(m,n)$-Kite graph, the graph obtained from $P_n$ by attaching $C_3$ in both the end edges of $P_n$, are $k$-Contra Harmonic Mean Graphs. The derived results are demonstrated by means of sufficient illustrations which provide better understanding. It is possible to investigate similar results for several other graphs.
REFERENCES