An Alexandroff Bitopological Space on Undirected Graphs

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ABSTRACT

All nodes (vertices) has finitely many adjacent nodes(vertices) in graph $G = (V, E)$ is called locally finite graph, where $V$ is referred as vertex (node) set and $E$ is referred as edge (arc) set. Alexandroff spaces became much more important field because of their use in digital topology. Alexandroff space is a topological space, in which arbitrary intersection of open sets is open (or arbitrary union of closed sets is closed) equivalently, we say that each singleton has minimal neighborhood base. The bitopological spaces that is the triple $(A, \tau_1, \tau_2)$ of a collection $A$ with two (arbitrary) topologies $\tau_1$ and $\tau_2$ on $A$. In this paper, we mean by a bitopological space $(V, \tau_G, \tau_{IG})$ is an Alexandroff bitopological space, satisfy the stronger condition namely, arbitrary intersection of members of $S_G$ and $S_{IG}$ are open in $\tau_G$ and $\tau_{IG}$ respectively on $V$, where $S_G$ is the sub basis for a graphic topology $\tau_G$ and $S_{IG}$ is the sub basis for a incident topology $\tau_{IG}$. Latter, we investigate some properties and characterization of this topological spaces. In particular, the separation axioms are studied. Our goal is to consider the fundamental steps toward analyzing some properties of locally finite graphs by their corresponding topology.

KEYWORDS: Locally finite graphs, Alexandroff space, bitopological space.

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1 INTRODUCTION:

Nowadays, the graphs structure has heaps of applications in our real life. Mathematical structure of the graphs are helpful from practical perspective once they abstractly represents by graphs. Collection of nodes may be connected in graphs is employed nowadays to review issues in economics, networks of communication, knowledge organization, procedure devices, physics, chemistry, sociology, linguistics and numberless alternative fields. Topology is that the one in every of the foremost necessary fields in mathematics. A set becomes a topological space. It is the geometry part that does not concern distance. The one topology is extended to bitopology with usual definition. Kelly was the first who developed the ideas of bitopological spaces that is the triple \((A, \tau_1, \tau_2)\) of a collection \(A\) with two (arbitrary) topologies \(\tau_1\) and \(\tau_2\) on \(A\). Baby Girija and Pilakkat introduced bitopological spaces associated with digraphs \(D = (V, E)\) employing a nonnegative real valued function \(P\) on \(V \times V\) known as quasi pseudo metric that develops two topologies on \(V\). And then Khalid Abdulkalek Abdu and Adem kilicman introduced Bitopological spaces on undirected graphs \(G = (V, E)\) by using two different sub basis families to come up with two topologies on \(V\).

In this paper, we tend to think about solely locally finite simple graphs to this Alexandroff bitopological space.

2 PRELIMINARIES:

The bitopological space \((A, \tau_1, \tau_2)\) is pair wise \(T_0\) if for each pair \((u,v)\) of distinct points of \(A\) there is either a \(\tau_1\) -open set containing \(u\) but not \(v\) or there exists a \(\tau_2\)-open set containing \(v\) but not \(u\). If for each pair of distinct points there exists a \(\tau_1\)-open set or \(\tau_2\)-open set containing one but the other, then \((A, \tau_1, \tau_2)\) is weakly pair wise \(T_0\). If for each pair of distinct points \(u, v\) there exists a \(\tau_1\)-open set \(D\) and a \(\tau_2\)-open set \(W\) such that \(u \in D, v \notin D\) and \(v \in W, u \notin W\) or \(u \in W, v \notin W\) and \(v \in D, u \notin D\) then \((A, \tau_1, \tau_2)\) is weakly pair wise \(T_1\). The bitopological space \((A, \tau_1, \tau_2)\) is weakly pair wise \(T_2\) if for each pair of distinct points \(u, v\) there exists a \(\tau_1\)-open set \(D\) and a \(\tau_2\)-open set \(W\) with \(D \cap W = \emptyset\) such that \(u \in D\) and \(v \in W\) or \(u \in W\) and \(v \in D\). If for each pair of distinct points \((u,v)\) there exists a \(\tau_1\)-open set \(D\) and a \(\tau_2\)-open set \(W\) with \(D \cap W = \emptyset\) such that \(u \in D\) and \(v \in W\) then \((A, \tau_1, \tau_2)\) is pair wise \(T_2\).

Definition 2.1

Let \(G = (V, E)\) be a (simple) graph without isolated vertex. Remember that \(A_x\) is the set of all
vertices adjacent to \( x \). It is clear that \( x \in A_y \) if and only if \( y \in A_x \) for all \( x, y \in V \) and \( x \notin A_x \) for all \( x \in V \). Define \( S_G \) as follows: \( S_G = \{ A_x | x \in V \} \). Since \( G \) has no isolated vertex we have \( V = \bigcup_{x \in V} A_x \). Hence \( S_G \) forms a sub basis for a topology \( \tau_G \) on \( V \) called graphic topology of \( G \).

**Note**
: Topologies of \( K_n \) and \( C_n \) are discrete, but the graphic topology of \( P_n \) is not discrete because the set contains two vertices of degree one is not open. Also graphic topology of \( K_{n,m} \) is equal to \( \{Q, V, A, B\} \) where \( A \) and \( B \) are partite sets of \( K_{n,m} \).

**Definition 2.2**

Let \( G = (V, E) \) be a (simple) graph without isolated vertex. Let \( I_e \) be the incidence vertices with the edge \( e \). Define \( S_{IG} \) as follows: \( S_{IG} = \{ I_e | e \in E \} \). Since \( G \) has no isolated vertex we have \( V = \bigcup_{e \in E} I_e \). So \( S_{IG} \) forms a subbasis for a topology \( \tau_{IG} \) on \( V \) called incidence topology of \( G \).

**Note**
: It is obvious that the incidence topologies of Cycle \( C_n, n \geq 3 \), the Complete graph \( K_n, n \geq 3 \) and the Complete bipartite graph \( K_{n,m}, n, m \geq 1 \) are discrete, but the incidence topology of the Path \( P_n \) is not discrete because \( P_n \) contains two vertices incident with one edge is not open.

**Proposition 2.3**

Suppose that \( G = (V, E) \) is a graph. Then \( (V, \tau_{IG}) \) is an Alexandroff space.

**Proposition 2.4**

Suppose that \( \tau_{IG} \) is the incidence topology of the graph \( G = (V, E) \). If \( d(v) \geq 2 \), then \( \{v\} \in \tau_{IG} \) for every \( v \in V \).

**Proposition 2.5**

Let \( G = (V, E) \) be a graph. If \( d(v) \geq 2 \) for all \( v \in V \), then \( \tau_{IG} \) is a discrete topology.

**Proposition 2.6**

In any graph \( G = (V, E) \), \( U_v = \bigcap_{e \in I_v} I_e \) for every \( v \in V \).

**Remark 2.7**

Let \( G = (V, E) \) be a graph, then \( I_v \) is the set of all edges incident with the vertex \( v \).

**Remark 2.8**

The Alexandroff topological space \( (X, \tau) \) is \( T_1 \) if and only if \( U_x = \{x\} \). It follows that \( (X, \tau) \) is discrete. Therefore, the incidence topology \( (V, \tau_{IG}) \) which is Alexandroff space \( T_1 \) if and only if it is
discrete. If \((V, \tau_G)\) is an Alexandroff space, then \((V, \tau IG)\) is \(T_0\) space if and only if \(U_u=U_v\) implies \(u=v\). This means \(U_u \neq U_v\) for all distinct pair of vertices \(u, v \in V\). The incidence topology is \(T_0\) if and only if \(I_u \neq I_v\) for every distinct pair of vertices \(u, v \in V\).

3 AN ALEXANDROFF BITOPOLOGICAL SPACE:

The graphic topology and the incidence topology on \(V\) forms the bitopological space \((V, \tau_G, \tau_{IG})\). This bitopological Space \((V, \tau_G, \tau_{IG})\) is called Alexandroff bitopological space if and only if arbitrary intersection of members of \(S_G\) and \(S_{IG}\) are open in \(\tau_G\) and \(\tau_{IG}\) respectively on \(V\).

Example 3.1

Let \(G = (V, E)\) be a simple graph as in figure such that

\[ V = \{v_1, v_2, v_3, v_4\} \quad \text{and} \quad E = \{e_1, e_2, e_3\} \]

![Graph diagram]

Then \(S_G = \{\{v_2\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_3\}\}\) and \(S_{IG} = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\}\) and \(\tau_G = \{\emptyset, V, \{v_2\}, \{v_1, v_3\}, \{v_1, v_2, v_3\}, \{v_3, v_4\}\}\) and \(\tau_{IG} = \{\emptyset, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\}\) Therefore \(\tau_G\) and \(\tau_{IG}\) on \(V\) give the Alexandroff bitopological space \((V, \tau_G, \tau_{IG})\).

Theorem 3.2

Let \(G = (V, E)\) be a locally finite graph. Then the bitopological space \((V, \tau_G, \tau_{IG})\) is Alexandroff bitopological space.

Proof:

First we have to prove that the arbitrary intersection of members of \(S_G\) is open in \(\tau_G\). If \(x \in \bigcap_{y \in V_1} A_y\), for some subset \(V_1\) of \(V\). This leads that \(x \in A_y\) for each \(y \in V_1\). Therefore \(y \in A_x\) for each \(y \in V_1\) and hence \(V_1 \subseteq A_x\). Since \(G\) is locally finite, \(A_x\) and so \(V_1\) are finite sets, this means that \(V_1\) is finite, then \(\bigcap_{y \in V_1} A_y\) is empty. But if \(V_1\) is finite, then \(\bigcap_{y \in V_1} A_y\) is the intersection of finitely many open sets and is open in \(\tau_G\).

Now to prove that the arbitrary intersection of members of \(S_{IG}\) is open in \(\tau_{IG}\). If \(v \in \bigcap_{e \in E} I_e\),
for some subset $E_1$ of $E$. This leads that $v \in I_e$ for each $e \in E_1$. Therefore $e \in I_v$ for each $e \in E_1$ and hence $E_1 \subseteq I_v$ and so $v \in \bigcap_{e \in E_1} I_e$. Since $G$ is locally finite, therefore $I_e$ and $E_1$ are finite sets. Then $\bigcap_{e \in E_1} I_e$ is the intersection of finitely many open sets and is open in $\tau_{IG}$. 

**Theorem 3.3**

The Alexandroff bitopological space $(V, \tau_G, \tau_{IG})$ of a graph $G = (V, E)$ is pair wise $T_0$.

**Proof:**

Let $(x, y)$ be the any distinct pair of vertices of $V$. Now there exists two cases

**Case:1** If the nodes $x$ and $y$ are adjacent, then by the definition of graphic topology $\tau_G$ there are two $\tau_G$-open sets $U_x$ and $U_y$ such that $U_x$ containing $x$ but not $y$ and $U_y$ containing $y$ but not $x$, where $U_x$ is the intersection of all open set containing $x$ is that the smallest open set and $U_y$ is the intersection of all open sets containing $y$ is that the smallest open set.

**Case:2** If the nodes $x$ and $y$ are not adjacent, then there exists two different edges $e_1, e_2 \in E$ such that $x$ incident with $e_1$ and $y$ incident with $e_2$. Then by the definition of incidence topology $\tau_G$, there are two $\tau_{IG}$ open sets $U_x$ and $U_y$ such that $U_x$ containing $x$ but not $y$ and $U_y$ containing $y$ but not $x$. From above cases, for each pair of distinct nodes $(x, y)$ of $V$, there is either a $\tau_G$ open set containing $x$ but not $y$ or there exists a $\tau_{IG}$ open set containing $y$ but not $x$. Hence the Alexandroff bitopological space $(V, \tau_G, \tau_{IG})$ is pair wise $T_0$.

**Theorem 3.4**

Let $G = (V, E)$ be a locally finite graph. Then the Alexandroff bitopological space $(V, \tau_G, \tau_{IG})$ is weakly pair wise $T_0$.

**Proof:**

Let $G = (V, E)$ be a simple graph. If $(V, \tau_{IG})$ is Alexandroff space then $(V, \tau_{IG})$ is $T_0$ space if and only if $U_x \neq U_y$, for every distinct pair of nodes $x, y \in V$, where $U_x$ is the intersection of all open set containing $x$ is that the smallest open set and $U_y$ is the intersection of all open sets containing $y$ is that the smallest open set. Suppose $U_x = U_y$, for every distinct pair of nodes $x, y \in V$ in $\tau_{IG}$, then which implies that $I_x = I_y$ for all distinct pair of nodes $x, y \in V$. This leads that $x$ and $y$ are adjacent nodes of degree one. Then by the definition of graphic topology $\{x\}, \{y\} \in \tau_G$. Therefore, for every pair of distinct nodes of $V$, there exists a $\tau_G$ open set or $\tau_{IG}$ open set containing one but not the other. Hence the Alexandroff bitopological space $(V, \tau_G, \tau_{IG})$ is weakly pair wise $T_0$. 
Theorem 3.5

The Alexandroff bitopological space \((V, \tau_G, \tau_{IG})\) of a graph \(G = (V, E)\) is weakly pair wise \(T_1\) if and only if \(U_x \neq U_y\) in \(\tau_G\) and \(\tau_{IG}\), for every distinct pair of nodes \(x, y \in V\).

Proof:
Let us assume Alexandroff bitopological space \((V, \tau_G, \tau_{IG})\) be a weakly pairwise \(T_1\). By contradiction, suppose that there exists a distinct pair of nodes \(x, y \in V\) such that \(U_x = U_y\) in \(\tau_G\) or \(\tau_{IG}\).

Case:1 If \(U_x = U_y\) in \(\tau_G\), which implies that \(A_x = A_y\) then \(\tau_G\) is not a \(T_0\) space. That is there is no \(\tau_G\) open set containing \(x\) but not \(y\) or containing \(y\) but not \(x\). Which is contradiction to Alexandroff bitopological space \((V, \tau_G, \tau_{IG})\) is weakly pair wise \(T_1\).

Case:2 If \(U_x = U_y\) in \(\tau_{IG}\), which implies that \(I_x = I_y\) then \(\tau_{IG}\) is not a \(T_0\) space. That is there is no \(\tau_{IG}\) open set containing \(x\) but not \(y\) or containing \(y\) but not \(x\). Which is contradiction to Alexandroff bitopological space \((V, \tau_G, \tau_{IG})\) is weakly pair wise \(T_1\).

Conversely, let us assume that \(U_x \neq U_y\) in \(\tau_G\) and \(\tau_{IG}\) for every distinct pair of nodes \(x, y \in V\). Now we have two cases,

Case:1 If the nodes \(x\) and \(y\) are adjacent then by the definition of graphic topology there are two \(\tau_G\) open sets \(U_x\) and \(U_y\) such that \(U_x\) containing \(x\) but not \(y\) and \(U_y\) containing \(y\) but not \(x\). From our assumption \(U_x \neq U_y\) in \(\tau_G\) for every distinct pair of nodes \(x, y \in V\) in \(\tau_{IG}\), so that \(\tau_{IG}\) is \(T_0\) space. That is there exists \(\tau_{IG}\) open set containing \(x\) but not \(y\) or containing \(y\) but not \(x\).

Case:2 The nodes \(x\) and \(y\) are not adjacent. This means that there exists two different edges \(e_1, e_2 \in E\) such that \(x\) incident with \(e_1\) and \(y\) incident with \(e_2\). Then by the definition of incidence topology \(\tau_{IG}\) there are two \(\tau_{IG}\) open sets such that \(U_x\) containing \(x\) but not \(y\) and \(U_y\) containing \(y\) but not \(x\). From our assumption \(U_x \neq U_y\) for every distinct pair of nodes \(x, y \in V\) in \(\tau_G\) is \(T_0\) space. That is there exists \(\tau_{IG}\) open set containing \(x\) but not \(y\) or containing \(y\) but not \(x\). From cases above, for each pair of distinct nodes \(x, y \in V\), there exists a \(\tau_{IG}\) open set \(D\) and \(\tau_{IG}\) open set \(W\) such either that \(x \in D, y \notin D\) and \(x \notin W, y \in W\) or \(x \in W, y \notin W\) and \(x \notin D, y \in D\). Hence the Alexandroff bitopological space \((V, \tau_G, \tau_{IG})\) of a graph \(G = (V, E)\) is weakly pair wise \(T_1\).
The Alexandroff bitopological space \((V, \tau_G, \tau_{IG})\) of a graph \(G = (V, E)\) is weakly pair wise \(T_1\) if and only if \(\tau_G\) and \(\tau_{IG}\) are discrete topologies.

**Proof:**

The Alexandroff bitopological space \((V, \tau_G, \tau_{IG})\) is pair wise \(T_1\) if and only if \(\tau_G\) and \(\tau_{IG}\) is \(T_1\), because pair wise \(T_1\) in each topology if and only if \(\tau_G\) and \(\tau_{IG}\) are discrete topologies. Since from [remark 2.8 \(\tau_G\) and \(\tau_{IG}\) are \(T_1\) spaces if and only if \(\tau_G\) and \(\tau_{IG}\) are discrete topologies.

**Theorem 3.7**

Let \(G = (V, E)\) be a locally finite graph. Then the Alexandroff bitopological space \((V, \tau_G, \tau_{IG})\) is pair wise \(T_2\) if and only if for every distinct pair of nodes \(x, y \in V\) such that \(U_x \neq U_y\) in \(\tau_G\) and \(\tau_{IG}\) and the length of any Path is \(P \geq 4\).

**Proof:**

Let us assume that the Alexandroff bitopological space \((V, \tau_G, \tau_{IG})\) is weakly pair wise \(T_2\). By contradiction, suppose that there exists a distinct pair of nodes \(x, y \in V\) such that \(U_x = U_y\) in \(\tau_G\) or \(\tau_{IG}\) or there is a path of length less than four between two distinct pendent nodes.

**Case 1:** \(U_x = U_y\) in \(\tau_G\) which implies that \(A_x = A_y\) in \(\tau_G\). Therefore \(\tau_G\) is not a \(T_0\) space. That is there is no \(\tau_G\) open set containing \(x\) but not \(y\) or containing \(y\) but not \(x\). Which is contradiction to Alexandroff bitopological space \((V, \tau_G, \tau_{IG})\) is weakly pair wise \(T_2\).

**Case 2:** \(U_x = U_y\) in \(\tau_{IG}\) which implies that \(I_x = I_y\), Then \(\tau_{IG}\) is not a \(T_0\) space. That is there is no \(\tau_{IG}\) open set containing \(x\) but not \(y\) or containing \(y\) but not \(x\). Which is contradiction to Alexandroff bitopological space \((V, \tau_G, \tau_{IG})\) is weakly pair wise \(T_2\).

**Case 3** Suppose that \(x\) and \(y\) are two distinct pendent nodes and \(P\) is the path of length less than four between \(x\) and \(y\). If the Path of length is one or two then \(U_x = U_y\) in \(\tau_G\) or \(\tau_{IG}\) respectively, which is contradiction with the assumption. Suppose the Path of length of the length is three, let the Path \(P = xe_1ue_2ve_3y\), where \(x, u, v, y \in V\) and \(e_1, e_2, e_3 \in E\). Then the open set in \(\tau_G\) that contain \(x\) and \(y\) are \(U_x\) and \(U_y\) respectively, such that \(x, v \in U_x\) and \(u, y \in U_y\). And also the open set in \(\tau_{IG}\) that contain \(x\) and \(y\) are \(U_x\) and \(U_y\) respectively, such that \(x, u \in U_x\) and \(v, y \in U_y\). Thus we have \(A_u \cap I_{e_1} = \emptyset\) and \(A_v \cap I_{e_3} = \emptyset\). So there is no \(\tau_G\) open set \(D\) and \(\tau_{IG}\) open set \(W\) such that \(x \in D\) and \(y \in W\) or \(x \in W\) and \(y \in D\), which is contradiction with the assumption. Since the Alexandroff bitopological space \((V, \tau_G, \tau_{IG})\) is weakly pair wise \(T_2\).
conversely, let us assume that $U_x \neq U_y$ for every distinct pair of nodes $x, y \in V$ in $\tau_G$ and $\tau_{IG}$ and the length of any path between any two distinct pendent nodes is at least four. Let us consider, any pair of distinct nodes $x, y \in V$, we have three cases.

**Case:1** $x$ and $y$ are adjacent nodes such that $d(x) = 1$ and $d(y) \geq 2$ then $y \in \tau_{IG}$. From the definition of graphic topology $U_y$ is an open set containing $y$ but not $x$. This implies that $A_y$ is open set containing $x$ and but not $y$. Hence $A_y$ is $\tau_G$ open set containing $x$ and $\{y\}$ is $\tau_{IG}$ open set containing $y$ such that $A_y \cap \{y\} = \emptyset$.

**Case:2** $x$ and $y$ are not adjacent nodes of degree at least two. We have $\{x\}, \{y\} \in \tau_{IG}$. From our assumption $U_x \neq U_y$ for every distinct pair of nodes $x, y \in V$ in $\tau_G$. This implies that $A_x \neq A_y$ for every distinct pair of nodes $x, y \in V$. Then $\tau_G$ is $T_0$ space. That is there exists a $\tau_G$ open set containing $x$ but not $y$ or containing $y$ but not $x$. Therefore, there exists a $\tau_G$ open set $D$ and $\tau_{IG}$ open set $W$ with $D \cap W = \emptyset$ such that $x \in D, y \in W$ or $x \in W$ and $y \in D$.

**Case:3** $x$ and $y$ are not adjacent nodes, such that $d(x) = 1$ and $d(y) \geq 2$. Suppose that $u \in V$ is a node adjacent with $x$. This means there exists an edge $e \in E$ such that $e = xu$. Now either $y$ adjacent with $u$ or $y$ is not adjacent with $u$.

a) If $y$ is adjacent with $u$ then $I_e = \{x, u\}$ is $\tau_G$ open set by the definition of incidence topology. Since $y$ is adjacent with $u$ from graphic topology $\tau_G$ we have $x, u \in U_y$. Hence $I_e$ is $\tau_{IG}$ open set containing $x$ and $U_y$ is $\tau_G$ open set containing $y$ such that $I_e \cap U_y = \emptyset$.

b) If $y$ is not adjacent with $u$ then $U_u$ is an open set containing $u$ but not $y$ in $\tau_G$. Which implies $A_u$ is an open set containing $x$ but not $y$ by the definition of graphic topology $\tau_G$. So we have $\{y\} \in \tau_{IG}$. Therefore, $A_u \in \tau_G$ open set containing $x$ and $\{y\}$ is $\tau_{IG}$ open set containing $y$ such that $A_u \cap \{y\} = \emptyset$.

**Case:4** $x$ and $y$ are not adjacent nodes such that $d(x) = 1$ and $d(y) = 1$. From assumption the length of any path between $x$ and $y$ is at least four. Let $P = xe_1ue_2ve_3e_4z$ be the path between $x$ and $y$ such that $x, u, v, y, z \in V$ and $e_1, e_2, e_3, e_4 \in E$ then by the definition of graphic topology and incidence topology $A_u$ and $I_{e_i}$ are $\tau_G$ open set and $\tau_{IG}$ open set respectively, $A_u \cap I_{e_i} = \emptyset$ such that $x \in A_u$ and $y \in I_{e_i}$. Form cases above, for each pair of nodes $x, y \in V$ there exists a $\tau_G$ open set $D$ and $\tau_{IG}$ open set $W$ with $D \cap W = \emptyset$ such that $x \in D, y \in W$ or $x \in W, y \in D$. Hence the Alexandroff bitopological space $(V, \tau_G, \tau_{IG})$ is weakly pair wise $T_2$. 
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