Square Multiplicative Labeling of Some Standard Graphs

M. Ganeshan* and M. S. Paulraj

Abstract

A graph is said to be a Square multiplicative labeling if there exists a bijection $g: V(G) \rightarrow \{1, 2, 3, \ldots, p\}$ such that the induced function $g^* : E(G) \rightarrow N$ given by $g^*(uv) = [g(u)]^2 \cdot [g(v)]^2$ for every $uv \in E(G)$ are all distinct. A graph which admits Square multiplicative labeling is called Square multiplicative graph. In this paper, we show that the almost bipartite graph and ladder graph admit square multiplicative labeling.

Keywords: Almost bipartite graph, Labeling, Ladder graph, Square multiplicative labeling.

AMS Subject Classification (2010): 05C78.

*Corresponding author

Mr. M. Ganeshan
Assistant professor, Department of Mathematics,
A.M.Jain college, Meenambakkam,
Chennai-600114. Tamilnadu, India.
Email: sivananthan21oct@gmail.com Cell. No: 9171070771.
INTRODUCTION

Labeling 4,7 of graph $G$ is the allocation of labels generally represented by integers to edges or vertices or both following certain conditions. There are several types of labeling techniques that have evolved from 1960s, one of the renowned labeling method is square multiplicative labeling2,3,5. In this paper we consider a simple, finite, connected and undirected graph.

Definition

$G$ is said to be a Square multiplicative labeling if there exists a bijection $g: V(G) \rightarrow \{1, 2, 3, \ldots, p\}$ such that the induced function $g^* : E(G) \rightarrow N$ given by $g^*(uv) = [g(u)]^2 [g(v)]^2$ for every $uv \in E(G)$ are all distinct. A graph which admits Square multiplicative labeling is called Square multiplicative graph.

Definition

An almost-bipartite graph1 is a non-bipartite graph with the property that the removal of a particular single edge renders the graph bipartite.

Definition

The ladder graph6 $L_n (n \geq 2)$ is the product graph $P_2 \times P_n$ which contains $2n$ vertices and $3n-2$ edges.

RESULTS

Theorem 1: The Almost bipartite graph $P_m + e$ is square multiplicative.

Proof

Consider the graph $G = P_m + e$, where $P_m$ is the path $u_1u_2u_3 \ldots \ldots u_m$. Let $V_1$ and $V_2$ be the bipartition of the vertex set of $G$.

Case-1: When $m$ is even , $e = u_1u_{m-1}$

$V_1 = \{u_1, u_3, \ldots, u_{m-1}\}$ and $V_2 = \{u_2, u_4, \ldots, u_m\}$.

Case-2: When $m$ is odd , $e = u_1u_m$

$V_1 = \{u_1, u_3, \ldots, u_m\}$ and $V_2 = \{u_2, u_4, \ldots, u_{m-1}\}$.

For both the cases we define $g: V(G) \rightarrow \{1, 2, 3, \ldots, m\}$ by $g(u_r) = r, 1 \leq r \leq m$.

The function $g$ induces a square multiplicative labeling on $G$.

For if $g^*$ be the induced function defined by $g^*: E \rightarrow N$ such that $g^*(u_r, u_s) = r^2 s^2$

Case-1: When $m$ is even , $e = u_1u_{m-1}$

Let $E = E_1 \cup E_2 \cup E_3$ where,

$E_1 = \{e_r | e_r = u_{2r-1}u_{2r}, 1 \leq r \leq \frac{m}{2}\}$,

$E_2 = \{e_r | e_r = u_{2r}u_{2r+1}, 1 \leq r \leq \frac{m}{2} - 1 \}$,
$E_3 = \{ e = u_1 u_{m-1} \}$.

To prove that $g^*$ is injective in $E$.

**Claim 1:** $g^*$ is injective in $E_1$.

Let $e_r, e_s \in E_1$

$g^*(e_r) = g^*(u_{2r-1} u_{2r})$

$= [g(u_{2r-1})]^2 [g(u_{2r})]^2$

$= (2r - 1)^2 (2r)^2$

$g^*(e_r) = 2^2 r^2 (2r - 1)^2$

$g^*(e_s) = g^*(u_{2s-1} u_{2s})$

$= [g(u_{2s-1})]^2 [g(u_{2s})]^2$

$= (2s - 1)^2 (2s)^2$

$g^*(e_s) = 2^2 s^2 (2s - 1)^2$

Hence for $r \neq s$, $g^*(e_r) \neq g^*(e_s)$

Hence $g^*$ is injective in $E_1$.

We find that all the labeling of edges in $E_1$ are multiples of $2^2$.

**Claim 2:** $g^*$ is injective in $E_2$.

Let $e_r, e_s \in E_2$

$g^*(e_r) = g^*(u_{2r} u_{2r+1})$

$= [g(u_{2r})]^2 [g(u_{2r+1})]^2$

$= (2r)^2 (2r + 1)^2$

$g^*(e_r) = 2^2 r^2 (2r + 1)^2$

$g^*(e_s) = g^*(u_{2s} u_{2s+1})$

$= [g(u_{2s})]^2 [g(u_{2s+1})]^2$

$= (2s)^2 (2s + 1)^2$

$g^*(e_s) = 2^2 s^2 (2s + 1)^2$

Hence for $r \neq s$, $g^*(e_r) \neq g^*(e_s)$

Hence $g^*$ is injective in $E_2$.

We observe that all the labelings of edges in $E_2$ are multiples of $2^2$.

**Claim 3:** $g^*$ is injective in $E_3$.

Let $e \in E_3$

$g^*(e) = g^*(u_1 u_{m-1})$

$= [g(u_1)]^2 [g(u_{m-1})]^2$

$g^*(e) = (1)^2 (m - 1)^2$
Since we have only one edge in $E_3$, $g^*$ is injective in $E_3$.

**Claim 4:** $g^*$ is injective among $E_1$, $E_2$ and $E_3$.

We notice that all the labeling of edges in $E_1$ and $E_2$ are multiple of $2^2$ and the edge label of $E_3$ is odd. Hence it is obvious that all the labelings of edges of $E_1$ and $E_2$ are distinct from the edge label of $E_3$. Now we have to show that labelings of edges in $E_1$ and $E_2$ are distinct.

**Claim 4.1:** $g^*$ is injective among $E_1$ and $E_2$.

Let $e_r \in E_1$, $e_s \in E_2$

\[
\begin{align*}
g^*(e_r) &= g^*(u_{2r-1}u_{2r}) \\
&= [g(u_{2r-1})]^2 [g(u_{2r})]^2 \\
&= (2r - 1)^2(2r)^2
\end{align*}
\]

\[
\begin{align*}
g^*(e_s) &= 2^2r^2(2r-1)^2
\end{align*}
\]

\[
\begin{align*}
g^*(e_r) &= g^*(u_{2s}u_{2s+1}) \\
&= [g(u_{2s})]^2 [g(u_{2s+1})]^2 \\
&= (2s)^2(2s+1)^2
\end{align*}
\]

\[
\begin{align*}
g^*(e_s) &= 2^2s^2(2s+1)^2
\end{align*}
\]

Hence for $r \neq s$, $g^*(e_r) \neq g^*(e_s)$

Hence $g^*$ is injective in $E_1$ and $E_2$.

$\Rightarrow$ All the edge labels in $E$ are distinct, when $m$ is even.

Hence $G$ admits square multiplicative labeling when $m$ is even.

Hence $G$ is a Square multiplicative Graph when $m$ is even.

**Case-2:** When $m$ is odd, $e = u_1u_m$

Let $E = E_1 \cup E_2 \cup E_3$ where,

$E_1 = \{e_r | e_r = u_{2r-1}u_{2r}, 1 \leq r \leq \frac{m+1}{2} - 1 \}$,

$E_2 = \{e_r | e_r = u_{2r}u_{2r+1}, 1 \leq r \leq \frac{m+1}{2} - 1 \}$,

$E_3 = \{e = u_1u_m \}$.

To prove that $g^*$ is injective in $E$.

**Claim 1:** $g^*$ is injective in $E_1$.

Let $e_r, e_s \in E_1$

\[
\begin{align*}
g^*(e_r) &= g^*(u_{2r-1}u_{2r}) \\
&= [g(u_{2r-1})]^2 [g(u_{2r})]^2 \\
&= (2r - 1)^2(2r)^2
\end{align*}
\]

\[
\begin{align*}
g^*(e_s) &= 2^2r^2(2r-1)^2
\end{align*}
\]

\[
\begin{align*}
g^*(e_r) &= g^*(u_{2s-1}u_{2s}) \\
&= [g(u_{2s-1})]^2 [g(u_{2s})]^2 \\
&= (2s - 1)^2(2s)^2
\end{align*}
\]

\[
\begin{align*}
g^*(e_s) &= 2^2s^2(2s-1)^2
\end{align*}
\]
\[ g(u_{2s-1})^2 [g(u_{2s})]^2 \]
\[ = (2s - 1)^2 (2s)^2 \]
\[ g^*(e_s) = 2^2 s^2 (2s - 1)^2 \]

Hence for \( r \neq s \), \( g^*(e_r) \neq g^*(e_s) \)

Hence \( g^* \) is injective in \( E_1 \).

We find that all the labeling of edges in \( E_1 \) are multiples of \( 2^2 \).

**Claim 2**: \( g^* \) is injective in \( E_2 \).

Let \( e_r, e_s \in E_2 \)
\[ g^*(e_r) = g^*(u_{2r} u_{2r+1}) \]
\[ = [g(u_{2r})]^2 [g(u_{2r+1})]^2 \]
\[ = (2r)^2 (2r + 1)^2 \]
\[ g^*(e_r) = 2^2 r^2 (2r + 1)^2 \]
\[ g^*(e_s) = g^*(u_{2s} u_{2s+1}) \]
\[ = [g(u_{2s})]^2 [g(u_{2s+1})]^2 \]
\[ = (2s)^2 (2s + 1)^2 \]
\[ g^*(e_s) = 2^2 s^2 (2s + 1)^2 \]

Hence for \( r \neq s \), \( g^*(e_r) \neq g^*(e_s) \). Hence \( g^* \) is injective in \( E_2 \).

We observe that all the labelings of edges in \( E_2 \) are multiples of \( 2^2 \).

**Claim 3**: \( g^* \) is injective in \( E_3 \).

Let \( e \in E_3 \)
\[ g^*(e) = g^*(u_1 u_m) \]
\[ = [g(u_1)]^2 [g(u_m)]^2 \]
\[ g^*(e) = (1)^2 (m)^2 \]

Since we have only one edge in \( E_3 \), \( g^* \) is injective in \( E_3 \).

**Claim 4**: \( g^* \) is injective among \( E_1, E_2 \) and \( E_3 \).

We notice that all the labeling of edges in \( E_1 \) and \( E_2 \) are multiples of \( 2^2 \) and the edge label of \( E_3 \) is odd. Hence it is obvious that all the labelings of edges of \( E_1 \) and \( E_2 \) are distinct from the edge label of \( E_3 \). Now we have to show that labelings of edges in \( E_1 \) and \( E_2 \) are distinct.

**Claim 4.1**: \( g^* \) is injective among \( E_1 \) and \( E_2 \).

Let \( e_r \in E_1, e_s \in E_2 \)
\[ g^*(e_r) = g^*(u_{2r-1} u_{2r}) \]
\[ = [g(u_{2r-1})]^2 [g(u_{2r})]^2 \]
\[ = (2r - 1)^2 (2r)^2 \]
\[ g^*(e_r) = 2^2 r^2 (2r - 1)^2 \]
\[ g^*(e_s) = g^*(u_{2s}u_{2s+1}) \]
\[ = [g(u_{2s})]^2 [g(u_{2s+1})]^2 \]
\[ = (2s)^2 (2s + 1)^2 \]
\[ g^*(e_r) = 2^2 s^2 (2s + 1)^2 \]

Hence for \( r \neq s \), \( g^*(e_r) \neq g^*(e_s) \)

Hence \( g^* \) is injective in \( E_1 \) and \( E_2 \).

\( \Rightarrow \) All the edge labels in \( E \) are distinct, when \( m \) is odd.

Hence \( G \) admits square multiplicative labeling when \( m \) is odd.

\( \Rightarrow G \) admits Square multiplicative labeling for both the cases.

Hence \( G \) is a Square multiplicative Graph.

**Figure 1: Square Multiplicative Labeling Of Almost Bipartite Graph \( P_8 + e \).**

**Figure 2: Square Multiplicative Labeling Of Almost Bipartite Graph \( P_7 + e \).**

**Theorem 2:** The Ladder graph \( L_n \) is square multiplicative.

**Proof:**

We have \( |V(L_n)| = 2n \) and \( |E(L_n)| = 3n - 2 \).

Let \( V_1 \) and \( V_2 \) be the bipartition of the vertex set \( V \) where

\[ V_1 = \{ v_1, v_2, \ldots, v_n \} \] and \( V_2 = \{ u_1, u_2, \ldots, u_n \} \).

Let \( E = E_1 \cup E_2 \cup E_3 \) where,

\[ E_1 = \{ e_r \mid e_r = v_r v_{r+1}, 1 \leq r \leq n - 1 \}, \]
\[ E_2 = \{ e_r \mid e_r = u_r u_{r+1}, 1 \leq r \leq n - 1 \}, \]
\[ E_3 = \{ e_r \mid e_r = v_r u_r, 1 \leq r \leq n \}. \]

Define \( g:V(G) \rightarrow \{1, 2, 3, \ldots, 2n\} \) by
\[ g(v_r) = 2r - 1, \ 1 \leq r \leq n, \]
\[ g(u_r) = 2r, \ 1 \leq r \leq n. \]

The function \( g \) induces a square multiplicative labeling on \( G \).

For if, \( g^* \) be the induced function defined by \( g^*: E \rightarrow N \) such that
\[ g^*(v_r, u_r) = (2r - 1)^2(2s)^2 \]

To prove that \( g^* \) is injective in \( E \).

**Claim 1:** \( g^* \) is injective in \( E_1 \).

Let \( e_r, e_s \in E_1 \)
\[ g^*(e_r) = g^*(v_r v_{r+1}) = [g(v_r)]^2 [g(v_{r+1})]^2 \]
\[ g^*(e_s) = (2r - 1)^2(2r + 1)^2 \]
\[ g^*(e_r) = g^*(v_s v_{s+1}) = [g(v_s)]^2 [g(v_{s+1})]^2 \]
\[ g^*(e_s) = (2s - 1)^2(2s + 1)^2 \]

Hence for \( r \neq s \), \( g^*(e_r) \neq g^*(e_s) \)

Hence \( g^* \) is injective in \( E_1 \)

**Claim 2:** \( g^* \) is injective in \( E_2 \).

Let \( e_r, e_s \in E_1 \)
\[ g^*(e_r) = g^*(u_r u_{r+1}) = [g(u_r)]^2 [g(u_{r+1})]^2 \]
\[ g^*(e_s) = (2r)^2(2(r + 1))^2 \]
\[ g^*(e_r) = (2)^4(r(r + 1))^2 \]
\[ g^*(e_s) = g^*(u_s u_{s+1}) = [g(u_s)]^2 [g(u_{s+1})]^2 \]
\[ g^*(e_r) = (2s)^2(2(s + 1))^2 \]
\[ g^*(e_s) = (2)^4(s(s + 1))^2 \]

Hence for \( r \neq s \), \( g^*(e_r) \neq g^*(e_s) \)

Hence \( g^* \) is injective in \( E_2 \)

We note that all the labelings of edges in \( E_2 \) are multiples of \( 2^4 \)

**Claim 3:** \( g^* \) is injective in \( E_3 \).

Let \( e_r, e_s \in E_3 \)
\[ g^*(e_r) = g^*(v_r u_r) = [g(v_r)]^2 [g(u_r)]^2 \]
\[(2r - 1)^2(2r)^2\]

\[g^*(e_r) = (2)^2(2r - 1)^2(r)^2\]

\[g^*(e_s) = g^*(v_s u_s)\]
\[= [g(v_s)]^2 [g(u_s)]^2\]
\[= (2s - 1)^2(2s)^2\]

\[g^*(e_r) = (2)^2(2s - 1)^2(s)^2\]

Hence for \(r \neq s\), \(g^*(e_r) \neq g^*(e_s)\)

Hence \(g^*\) is injective in \(E_3\)

We note that all the labelings of edges in \(E_3\) are multiples of \(2^2\).

**Claim 4**: \(g^*\) is injective among \(E_1, E_2\) and \(E_3\).

We note that all the labelings of edges in \(E_2\) are multiples of \(2^4\).

Hence it is very clear that all the edge labels of \(E_2\) are distinct from the edge labels of \(E_1\) and \(E_3\).

Also we find that the edge labels of \(E_3\) are multiples of \(2^2\).

Hence all the edge labels of \(E_3\) are distinct from the edge labels of \(E_1\).

so all the edge labels in \(E\) are distinct. Hence \(L_n\) admits square multiplicative labeling.

\[\therefore\] The ladder graph \(L_n\) is square multiplicative.

![Figure 3:Square Multiplicative Labeling Of Ladder Graph L_7](image)

**CONCLUSION**

In this paper, the admittance of square multiplicative labeling to a variety of graphs namely almost bipartite graph and ladder graph has been discussed.

For almost bipartite graph two cases are discussed. This concept can be extended for different non-regular graphs.
REFERENCES


