On The Upper Open Geodetic Domination Number of a Graph

Vijimon Moni.V¹ and Robinson Chellathurai.S²

Register Number-12357,
¹St. Xavier’s Catholic College of Engineering, Chunkankadai-629 013
²Scott Christian College, Nagercoil-629 003,India.
Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012,
Tamil Nadu, India

ABSTRACT

Let \( G = (V,E) \) be a connected graph of order \( n \). A set \( S \subseteq V(G) \) is called an open geodetic dominating set of \( G \) if \( S \) is both open geodetic set and dominating set of \( G \). The minimum cardinality of an open geodetic dominating set of \( G \) is called the open geodetic domination number of \( G \) and is denoted by \( \gamma_{og}(G) \). An open geodetic dominating set of minimum cardinality is called \( \gamma_{og}^- \) set of \( G \).

An open geodetic dominating set \( S \) in a connected graph \( G \) is called a minimal open geodetic dominating set of \( G \) if no proper subset of \( S \) is an open geodetic dominating set of \( G \). The maximum cardinality of a minimal open geodetic domination set of \( G \) is the upper open geodetic domination number of \( G \) and is denoted by \( \gamma_{og}^+(G) \). A minimal open geodetic dominating set of cardinality \( \gamma_{og}^+(G) \) is called a \( \gamma_{og}^+ \) set of \( G \). The upper open geodetic dominating number of certain classes of graph are determined. Some general properties satisfied by this concept are studied. For any positive integers \( a \) and \( b \) with \( 2 \leq a \leq b \), there exists a connected graph \( G \) with \( \gamma_{og}(G) = a \) and \( \gamma_{og}^+(G) = b \).

KEYWORDS: Open geodetic number, Open geodetic domination number, upper open geodetic dominating number.

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*Corresponding author

Vijimon Moni.V

Register Number-12357, Department of Mathematics
St. Xaviers Catholic College of Engineering,
Chunkankadai-629 003, Tamilnadu. India.
Email:vijimon1983@gmail.com.
Mobile: 8946046108.
INTRODUCTION

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $n$ and $m$ respectively. For basic graph theoretic terminology, we refer to Harary\textsuperscript{10}. The $\text{distanced}(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u − v$ path in $G$. An $u − v$ path of length $d(u, v)$ is called an $u − v$ geodesic. A vertex $x$ is said to lie on a $u − v$ geodesic $P$ if $x$ is a vertex of $P$ including the vertices $u$ and $v$. The closed interval consists of $x, y$ and all vertices lying on some $x − y$ geodesic of $G^1$. For a non-empty set $S \subseteq V(G)$, the set $I[S] = \bigcup_{x, y \in S} I[x, y]$ is the closure of $S$. A set $S \subseteq V(G)$ is called a geodetic set if $I[S] = V(G)$. Thus every vertex of $G$ is contained in a geodesic joining some pair of vertices in $S$. The minimum cardinality of a geodetic set of $G$ is called the geodetic number of $G$ and is denoted by $g(G)$. A geodetic set of minimum cardinality is called $g$-set of $G^{2, 4, 5, 6}$. $N(v) = \{u \in V(G) : uv \in E(G)\}$ is called the neighborhood of the vertex $v$ in $G$. A vertex $v$ is an extreme vertex of a graph $G$ if $\langle N(v) \rangle$ is complete. A set of vertices $D$ in a graph $G$ is a dominating set if each vertex of $G$ is dominated by some vertex of $D$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set of $G^{3, 7}$. If $e = \{u, v\}$ is an edge of a graph $G$ with $d(u) = 1$ and $d(v) > 1$, then we call $e$ a pendant edge, $u$ a leaf and $v$ a support vertex. Let $L(G)$ be the set of all leaves of a graph $G$. For any connected graph $G$, a vertex $v \in V(G)$ is called a cut vertex of $G$ if $V − v$ is no longer connected. A set of vertices $S$ in $G$ is called a geodetic dominating set if $S$ is both a geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set of $G$ is its geodetic domination number and is denoted by $\gamma_g(G)$. A geodetic dominating set of size $\gamma_g(G)$ is said to be a $\gamma_g$-set of $G^{9, 12}$. A set $S$ of vertices of a connected graph $G$ is an open geodetic set if for each vertex $v$ in $G$ either $v$ is an extreme vertex of $G$ and $v \in S$ or $v$ is an internal vertex of a $x − y$ geodesic for some $x, y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the open geodetic number and is denoted by $og(G)$\textsuperscript{14}. A set $S \subseteq V(G)$ is called an open geodetic dominating set of a connected graph $G$ if $S$ is both open geodetic set and dominating set of $G$. The minimum cardinality of an open geodetic dominating set of $G$ is called open geodetic domination number of $G$ and is denoted by $\gamma_{og}(G)$\textsuperscript{13}. An open geodetic dominating set of minimum cardinality is called a $\gamma_{og}$-set of $G$.

For a cut vertex $v$ in a connected graph $G$ and the component $H$ of $G − v$, the subgraph $H$ and the vertex $v$ together with all edges joining $v$ to $V(H)$ is called a branch of $G$ at $v$. The middle graph of a graph $G = (V, E)$ is the graph $M(G) = (V \cup E, E')$, Where $uv \in E'$ if and only if either $u$ is a vertex of $G$ and $v$ is an edge of $G$ containing $u$, or $u$ and $v$ are edges in $G$ having a vertex in common.

The following theorem is used in sequel.
Theorem 1.1[13]. Let $G$ be a connected graph of order $n$. Then

i. every open geodetic dominating set of a graph $G$ contains its extreme vertices.

ii. every end vertex belongs to every open geodetic dominating set of $G$.

iii. if the set $S$ of extreme vertices of $G$ is a open geodetic dominating set of $G$, then $S$ is the unique minimum open geodetic dominating set of $G$ and $\gamma_{og}(G) = |S|$.

THE UPPER OPEN GEODETIC DOMINATION NUMBER OF A GRAPH

Definition 2.1. An open geodetic dominating set $S$ in a connected graph $G$ is called a minimal open geodetic dominating set of $G$ if no proper subset of $S$ is an open geodetic dominating set of $G$. The maximum cardinality of a minimal open geodetic dominating set of $G$ is the upper open geodetic set domination number of $G$ and is denoted by $\gamma_{og}^+(G)$. A minimal open geodetic dominating set of cardinality $\gamma_{og}^+(G)$ is called a $\gamma_{og}^+$-set of $G$.

Example 2.2. For the graph $G$ given in Figure 1, $S_1 = \{v_1, v_2, v_3, v_9\}$ and $S_2 = \{v_1, v_2, v_3, v_5, v_7, v_9\}$ are open geodetic dominating sets of $G$. It is clear that no proper subsets of $S_1$ and $S_2$ are open geodetic dominating sets of $G$ and so $S_1$ and $S_2$ are minimal open geodetic dominating sets of $G$. Hence $\gamma_{og}(G) = 5$ and $\gamma_{og}^+(G) = 6$. It is clear that there is no minimal open geodetic dominating set of cardinality greater than 6. Therefore

![Figure 1](image-url)

$\gamma_{og}^+(G) = 6$.

Theorem 2.3. Let $G$ be a connected graph of order $n$. Then

(i) every minimal open geodetic dominating set of a graph $G$ contains its extreme vertices. (ii) every end vertex belongs to every minimal open geodetic dominating set of $G$.

(iii) if $G$ has the unique minimal open geodetic dominating set, then $\gamma_{og}(G) = \gamma_{og}^+(G)$.

Proof. (i) Since every minimal open geodetic dominating set of connected graph $G$ is a open geodetic dominating set of $G$, by Theorem 1.1, (i) and (ii) follows immediately.

(iii) Let $S$ be unique minimal open geodetic dominating set of a connected graph $G$. Then it is clear that $\gamma_{og}(G) = |S|$ and $\gamma_{og}^+(G) = |S|$. Hence $\gamma_{og}(G) = \gamma_{og}^+(G)$.
Theorem 2.4. For the complete graph $G = K_2$, $\gamma_{og}^+(G) = n$.  

Proof. Since every vertex of $G$ is an extreme vertex, then by Theorem 2.3(i) $\gamma_{og}^+(G) = n$.  

Theorem 2.5. If a connected graph $G$ has $m$ extreme vertices, then $\gamma_{og}^+(G) \geq m$.  

Proof. As every minimal open geodetic dominating set of a connected graph $G$ contains its extreme vertices, by Theorem 2.3(i) $\gamma_{og}^+(G) \geq m$.  

Theorem 2.6. Let $M(G)$ be the middle graph of a connected graph $G$ of order $n$. Then $\gamma_{og}(M(G)) = \gamma_{og}^+(M(G)) = n$.  

Proof. Let $M(G)$ be the middle graph of a connected graph $G$ of order $n$. Then it is clear that set of extreme vertices of $M(G)$ is $V(G)$. It is easily verified that $V(G)$ is the unique minimal open geodetic dominating set of $M(G)$. Therefore, by Theorem 2.3(iii) $\gamma_{og}(M(G)) = \gamma_{og}^+(M(G)) = n$.  

Theorem 2.7. Let $G$ be a connected graph of order $n$, $2 \leq \gamma_{og}(G) \leq \gamma_{og}^+(G) \leq n$.  

Proof. Since every open geodetic dominating set needs at least two vertices, Therefore $\gamma_{og}(G) \geq 2$. Since every minimal open geodetic dominating set is a open geodetic dominating set of $G$, $\gamma_{og}(G) \leq \gamma_{og}^+(G)$. Also since the set of all vertices of $G$ is an open geodetic dominating set of $G, \gamma_{og}^+(G) \leq n$. Hence $2 \leq \gamma_{og}(G) \leq \gamma_{og}^+(G) \leq n$.  

Remark 2.8. The bounds in Theorem 2.7 are sharp. For the path $G = P_2, \gamma_{og}(G) = 2$. For the star $G = K_{1,n-1}, \gamma_{og}(G) = \gamma_{og}^+(G) = n - 1$. For the complete graph, $G = K_n, \gamma_{og}(G) = \gamma_{og}^+(G) = n$. Also the bounds in Theorem 2.7 are strict. For the graph $G$ given in Figure 2, $\gamma_{og}(G) = 7, \gamma_{og}^+(G) = 8$ and $n = 11$. Thus $2 \leq \gamma_{og}(G) \leq \gamma_{og}^+(G) \leq n$.  

Theorem 2.9. For the connected graph $G \gamma_{og}^+(G) = 2$if and only if $\gamma_{og}^+(G) = 2$.  

Proof. If $\gamma_{og}^+(G) = 2$, then by Theorem 2.7, $\gamma_{og}(G) = 2$. Conversely, let $\gamma_{og}(G) = 2$. Then $G$ contains two extreme vertices $u$ and $v$ such that $S = \{u, v\}$ is the uniqueminimum $\gamma_{og}^{-}$-set of $G$.  

![Figure 2](image-url)
Since $S$ is subset of every open geodetic dominating set it follows that $S = \{u,v\}$ is the unique minimal open geodetic dominating set of $G$, so that $\gamma_{og}(G) = 2$.

**Theorem 2.10.** Let $G$ be a connected graph of order $n$. If $\gamma_{og}(G) = n$, then and only if $\gamma_{og}(G) = n$.

**Proof.** If $\gamma_{og}(G) = n$, then by Theorem 2.7, $\gamma_{og}(G) = n$. Conversely, let $\gamma_{og}(G) = n$. Then $S = V(G)$ is the unique minimal open geodetic dominating set of $G$. Hence it follows that $S$ is the unique minimum open geodetic dominating set of $G$, so that $\gamma_{og}(G) = n$.

**Theorem 2.11.** Let $G$ be a connected graph of order $n$. If $\gamma_{og}(G) = n - 1$, then $\gamma_{og}(G) = n - 1$.

**Proof.** Let $\gamma_{og}(G) = n - 1$. Then by Theorem 2.7, $\gamma_{og}(G) = n$ or $n - 1$. If $\gamma_{og}(G) = n$, then by Theorem 2.10, $\gamma_{og}(G) = n$, which is a contradiction. Therefore $\gamma_{og}(G) = n - 1$.

**Theorem 2.12.** For the complete Bipartite graph $G = K_{m,n}$ with $2 \leq m \leq n$, $\gamma_{og}(G) = 4$.

**Proof.** Let $G = K_{m,n}$. Let $X = \{u_1, u_2, \ldots, u_m\}$ and $Y = \{v_1, v_2, \ldots, v_n\}$ be the partite sets of $G$. Let $S = \{u_i, u_j, v_r, v_s\}$. Then $S$ is a minimal open geodetic dominating set of $G$ and so $\gamma_{og}(G) \geq 4$. We show that $\gamma_{og}(G) = 4$. If not, let $\gamma_{og}(G) \geq 5$. Then there exists a minimal open geodetic dominating set $S'$ such that $|S'| \geq 5$. If $S' \subseteq X$, then $S'$ is not a open geodetic dominating set of $G$, which is a contradiction. If $S' \subseteq Y$, then $S'$ is not a open geodetic dominating set of $G$, which is a contradiction. Therefore, $S' \subseteq X \cup Y$. Let $S' = S_1 \cup S_2$, where $S_1 \subseteq X$ and $S_2 \subseteq Y$. Then $|S_1| \geq 2$ and $|S_2| \geq 2$ since $|S'| \geq 5$, either $S_1$ or $S_2$ contains at least three vertices, without loss of generality let us assume that $|S_1| \geq 3$. Let $x, y, z \in S_1$ and $v \in S_2$. Then $x, y, z, u, v \in S'$. Let $S'' = S' - \{x\}$. Which is a contradiction to $S'$ is a minimal open geodetic dominating set of $G$. Let $S'' = S' - \{x\}$. Then $S''$ is a open geodetic dominating set of $G$ such that $S'' \subset S'$ which is a contradiction to $S'$ is a minimal open geodetic dominating set of $G$. Therefore $\gamma_{og}(G) = 4$.

**Theorem 2.13.** For any connected non-complete graph $G$ of order $n$, then $\gamma_{og}(G) \leq n - \delta(G)$.

**Proof.** Let $S$ be a upper open geodetic dominating set of a non-complete connected graph $G$ order $n$. Then $\gamma_{og}(G) = |S|$. We show that $|S| \leq n - \delta(G)$. Let $v \in S$. Assume that $v$ is adjacent to $m$ distinct vertices in $S$. Since $deg(v) > \delta(G)$, $v$ must be adjacent to at least $\delta(G) - m$ vertices in $V(G) - S$ and so $|V(G) - S| > \delta(G) - m$. If $m = 0$, then $|V(G) - S| \geq \delta(G)$, that is $|S| \leq |V(G)| - \delta(G) = n - \delta(G)$. If $m > 0$, then the $m$ distinct vertices belong to $N[S]$ and do not lie on a geodesic joining any pair of vertices of $S$. Since $S$ is a minimal open geodetic dominating set of $G$, $|V(G) - S| \geq (\delta(G) - m) + m = \delta(G)$. Hence $|S| \leq n - \delta(G)$. Therefore $\gamma_{og}(G) \leq n - \delta(G)$.
Remark 2.14. The bounds in Theorem 2.13 are sharp. For the graph $G = K_{1,n-1}$ of order $n$. It is clear that $\delta(G) = 1, n - \delta(G) = n - 1$ and $\gamma^+_o(G) = n - 1$. Thus $\gamma^+_o(G) = n - \delta(G)$. The bounds in Theorem 2.13 can be strict. For the graph $G$ in Figure 3, $\delta(G) = 1, \gamma^+_o(G) = 4, n = 6, n - \delta(G) = 5$. Thus $\gamma^+_o(G) < n - \delta(G)$.

![Figure 3](image_url)

Theorem 2.15. Let $G$ be a connected graph of order $n$ and $u \in V(G)$. If deg$(u) = 1$, then $\gamma^+_o(G - u) \leq \gamma^+_o(G)$.

Proof. Let $u \in V(G)$ and deg$(u) = 1$. Let $S$ be a minimal open geodetic dominating set of $G - u$ with maximum cardinality, so $\gamma^+_o(G - u) = |S|$. Since deg$(u) = 1, u$ is an end vertex and $u$ is adjacent to exactly one vertex, say $v$. By Theorem 2.3 every minimal open geodetic dominating set of $G$ contains $u$. We consider two cases.

Case(i): Let $v \in S$. Since $S$ is an open geodetic dominating set of $G - u$, there exists a vertex $w \in V(G - u)$ such that $w \in I[v, x] \subseteq I[S], w \in N[S], v, x \in I[S]$ and $d(v, x) \leq 3$. If $d(v, x) = 3$, then consider the set $S' = (S - \{v\}) \cup \{u, w\}$. If $d(v, x) \leq 2$ then consider the set $S' = (S - \{v\}) \cup \{u\}$. It is straightforward to verify that $S'$ is a minimal open geodetic dominating set of $G$. So that $\gamma^+_o(G - u) = |S| \leq |S'| \leq \gamma^+_o(G)$.

Case(ii): Let $v \not\in S$. Then consider the set $S' = S \cup \{u\}$. It is straightforward to verify that $S'$ is a minimal open geodetic dominating set of $G$. So that $\gamma^+_o(G - u) = |S| < |S'| \leq \gamma^+_o(G)$. Hence in both the cases, $\gamma^+_o(G - u) \leq \gamma^+_o(G)$.

Remark 2.16. The bounds in Theorem 2.15 are sharp. For the graph $G = P_4$, let $u$ be an end vertex of $G$. It is clear that $\gamma^+_o(G - u) = 2$ and $\gamma^+_o(G) = 2$. Hence $\gamma^+_o(G - u) = \gamma^+_o(G)$. The bounds in Theorem 2.16 can be strict. For the graph $G$ in Figure 4, $\gamma^+_o(G - u) = 3$ and $\gamma^+_o(G) = 4$. Hence $\gamma^+_o(G - u) < \gamma^+_o(G)$.  

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Remark 2.17. The converse of the Theorem 2.15 is need not true. For the complete graph $K_n$, it is clear that $\gamma_{og}^+(K_n) = n$, $\gamma_{og}^+(K_n - u) = n - 1$ and $\deg(u) = n - 1$ for every $u \in V(K_n)$. Hence $\gamma_{og}^+(K_n - u) < \gamma_{og}^+(K_n)$ but $\deg(u) \neq 1$.

Remark 2.18. Theorem 2.15 is not true if $\deg(u) \neq 1$. For the graph $G = P_5$, given in Figure 5, $\gamma_{og}^+(G) = 3$, $\gamma_{og}^+(G - u) = 4$ and $\deg(u) = 2 \neq 1$. Thus $\gamma_{og}^+(G - u) \neq \gamma_{og}^+(G)$.

Theorem 2.19. For any non-trivial tree $T$ with $n \geq 3$, there exists a vertex $v \in V(T)$ such that $\gamma_{og}^+(T - v) = \gamma_{og}^+(T)$.

Proof. Let $T$ be any non-trivial tree with $n \geq 3$. It can be verified that the result is true for $n = 3$. Since if $n = 3$ then $T = P_3$. Now consider the case that $n > 3$. Since $T$ has at least one vertex with degree greater than or equal to 2, there exists a vertex $v \in V(T)$ with $\deg(v) \geq 2$ such that $v$ is adjacent to at least one leaf and at most one non-leaf. If there exists a vertex $v$ such that $v$ is adjacent to at least one leaf and no non-leaf then it is clear that $T = K_{1,n-1}$ and $v$ is the support vertex. So that $\gamma_{og}^+(T - v) = n - 1 = \gamma_{og}^+(T)$. If there does not exist a vertex $v$ such that $v$ is adjacent to exactly one leaf, then it is clear that $v$ is adjacent to two or more leaves. Assume that $v$ is adjacent to exactly one non-leaf. By Theorem 2.3, every minimal opengeodetic dominating set of $T$ contains its leaves. So it is clear that $\gamma_{og}^+(T - v) = \gamma_{og}^+(T)$. If there exists a vertex $v$ such that $v$ is adjacent to exactly one leaf $u$ and one non-leaf, then $\deg(u) = 1$ and $\deg(v) = 2$. Let $T' = T - v - u$. Since $\deg(u) = 1$, By Theorem 2.16, $\gamma_{og}^+(T - v) \leq \gamma_{og}^+(T)$. Hence, $\gamma_{og}^+(T') \leq \gamma_{og}^+(T - u) \leq \gamma_{og}^+(T)$. However, we have $\gamma_{og}^+(T') > \gamma_{og}^+(T) - 1$. If $\gamma_{og}^+(T') = \gamma_{og}^+(T) - 1$, then $\gamma_{og}^+(T) = \gamma_{og}^+(T - u)$. If $\gamma_{og}^+(T') > \gamma_{og}^+(T) - 1$, then...
\[ y_{og}^+(T') = y_{og}^+(T) = y_{og}^+(T - u). \] Hence there exists a vertex \( v \in V(T) \) such that \( y_{og}^+(T - v) = y_{og}^+(T) \). ■

**Remark 2.20.** Theorem 2.19 is not true for any graph \( G \). For the complete graph \( K_n \).

\[ y_{og}^+(K_n - v) \neq y_{og}^+(K_n) \] for every \( v \in V(K_n) \).

**Theorem 2.21.** Let \( G \) be a connected graph of order \( n \). If \( G' \) is a graph obtained by adding \( k \), where \( 1 \leq k \leq n \), end edges to a graph \( G \), then \( y_{og}^+(G) \leq y_{og}^+(G') \leq y_{og}^+(G) + k \).

**Proof.** Let \( G \) be a connected graph of order \( n \) and let \( G' \) be a connected graph obtained from \( G \) by adding \( k \) end edges \( u_i v_i (1 \leq i \leq k) \), where each \( u_i \in V(G) \) and \( v_i \notin V(G) \). First we show that \( y_{og}^+(G) \leq y_{og}^+(G') \). Let \( S \) be a \( y_{og}^+ \)-set of \( G \). So \( y_{og}^+G) = |S| \). We now consider three cases.

**Case(i):** Let \( u_i \in S \) for all \( i \). Then let \( S' = S \cup \{v_1, v_2, \ldots, v_k\} \). Since each \( v_i \notin V(G) \) is an end vertex of \( G' \) and \( u_i \notin S, v_i \notin I[S] \) and \( v_i \notin N[S] \), \( S' \) is a minimal open geodetic dominating set of \( G' \). Therefore \( y_{og}^+(G) = |S| < |S'| \leq y_{og}^+(G') \).

**Case(ii):** Let \( u_i \in S \) for some \( i, 1 \leq i \leq k \). Since \( S \) is an open geodetic dominating set of \( G \), there exists a vertex \( v \notin S \) such that \( v \in I[u_i, x] \subseteq I[S], v \notin N[S] \) and \( d(u_i, x) \leq 3 \) for some \( x \in S \). If \( d(u_i, x) = 3 \), then consider the set \( S' = (S - \{u_i\}) \cup \{v_i, v\} \). If \( d(u_i, x) \leq 2 \), then consider the set \( S' = (S - \{u_i\}) \cup \{v_i\} \). It is easily verified that \( S' \) is a minimal open geodetic dominating set of \( G' \). Therefore \( y_{og}^+(G) = S \leq |S'| \leq y_{og}^+(G') \).

**Case(iii):** Let \( u_i \in S \) for all \( i, 1 \leq i \leq k \). Then by the similar argument as in case(ii), we can prove that \( y_{og}^+(G) \leq y_{og}^+(G') \). Next, we show that \( y_{og}^+(G') \leq y_{og}^+(G) + k \). Let \( S \subseteq V(G) \) and let \( S' = S \cup \{v_1, v_2, \ldots, v_k\} \) be a minimal open geodetic dominating set of \( G' \) with maximumcardinalityso that \( y_{og}^+(G') = |S'| = |S| + k \). Since \( S' \) is a minimal open geodetic dominating set of \( G' \), \( u_i \notin S \) for all \( i, 1 \leq i \leq k \). We show that \( S \) is a minimal open geodetic dominating set of \( G \). If \( u_i \in I[S] \) and \( u_i \notin N[S] \) for all \( u_i \in V(G) \), then \( S \) is a minimal open geodetic dominating set of \( G \). If, then there exists a vertex \( u_i \notin V(G) \) such that \( u_i \notin I[S] \) or \( u_i \notin N[S] \). Then the set \( S \cup \{u_i\} \) is a minimal open geodetic dominating set of \( G \). Hence \( y_{og}^+(G') = |S| + k \leq y_{og}^+(G) + k \).

**Theorem 2.22.** For any two integer \( a \) and \( n \) with \( 2 \leq a \leq n \), there exists a connected graph \( G \) with \( y_{og}^+(G) = a \) and \( |V(G)| = n \).

**Proof.** It can be easily verified that the result is true for \( 2 \leq n \leq 3 \). If \( n = 2 \), then \( G = K_2 \) and if \( n = 3 \), then \( G \) is either \( P_3 \) or \( K_3 \). For \( n \geq 4 \). If \( a = n \), then \( G = K_n \) and if \( a = n - 1 \), then \( G = K_{n-1} \). For \( a \leq n - 2 \). Let \( P: x, y, z \) be a path on three vertices. Let \( G \) be a graph obtained from \( P \) by adding new vertices \( z_1, z_2, \ldots, z_{a-3}, v_1, v_2, \ldots, v_{n-a} \) and joining each \( z_i (1 \leq i \leq a - 3) \) with \( z \), and
joining each \( v_i \) \( (1 \leq i \leq n - a) \) with \( x \) and \( z \). The graph \( G \) is shown in Figure 6. Let \( S = \{z_1, z_2, ..., z_{a-3}\} \). Then by Theorem 1.1 (i) \( S \) is a subset of every open geodetic dominating set. It is easily verified that \( S \cup \{u\} \), and \( S \cup \{u, v\} \) is not an open geodetic dominating set of \( G \) and so \( \gamma_{og}(G) \geq a \). Now \( S' = S \cup \{x\} \cup \{y, v_i\} \) \( (1 \leq i \leq n - a) \) or \( S' = S \cup \{x\} \cup \{v_i, v_j\} \) \( (1 \leq i, j \leq n - a) \) is a minimal open geodetic dominating set of \( G \) and so \( \gamma_{og}(G) \geq a \). We prove that \( \gamma_{og}(G) = a \). If not, suppose that \( \gamma_{og}(G) > a \). Then there exists a minimal open geodetic dominating set of \( S'' \) with \( |S''| \leq a + 1 \). Then \( S'' \) contains at least two \( v_i \) \( (1 \leq i \leq n - a) \). Now \( v_i \) must lie on \( I[x, z_j] \) for \( (1 \leq i \leq n - a) \) and \( (1 \leq j \leq a - 3) \). Then \( x \) must belong to \( S'' \). Then it follows that \( S' \subset S'' \), which is a contradiction to \( S'' \) is a minimal open geodetic dominating set of \( G \). Therefore \( \gamma_{og}(G) = a \).

\[ \]

**Theorem 2.23.** For any two integer \( a \) and \( b \) with \( 2 \leq a \leq b \), there exists an aconnected graph \( G \) with \( \gamma_{og}(G) = a \) and \( \gamma_{og}(G) = b \).

**Proof.** It can be easily verified that the result is true for \( 2 = a = b \). Consider the graph \( G = K_n \). It is clear that \( \gamma_{og}(K_2) = 2 \) and \( \gamma_{og}(K_2) = 2 \). If \( 2 < a = b \), then consider the graph \( G = K_n \) \( (n > 2) \). It is clear that \( \gamma_{og}(K_n) = \gamma_{og}(K_n) = n \). If \( 2 < a = b \), then consider the graph \( G = K_{1,n} \). It is clear that \( \gamma_{og}(K_{1,n}) = \gamma_{og}(K_{1,n}) = n - 1 \). Now we consider \( 2 < a < b \). Let \( P : x, u, v, w, t \) be a path on five vertices. Let \( H \) be a graph obtained from \( P \) by adding new vertices \( z_1, z_2, ..., z_{a-4} \) and joining each \( z_i \) \( (1 \leq i \leq a - 4) \) with \( u \). Let \( G \) be a graph obtained from \( H \) by adding new vertices \( y, s, v_1, v_2, ..., v_{b-a+1} \) and joining each \( v_i \) \( (1 \leq i \leq b - a + 1) \) with \( x \) and \( y \) and joint \( s \) with \( y \) and \( t \), the graph \( G \) is shown in Figure 7. First we show that \( \gamma_{og}(G) = a \). Let \( Z = \{z_1, z_2, ..., z_{a-4}\} \) be the set of all endvertices of \( G \). By Theorem 1.1 (i) \( Z \) is a subset of every open geodetic dominating set of \( G \). It is easily verified.
that $Z$ is not a open geodetic dominating set of $G$. It is easily verified that $Z \cup \{x_1\}$ or $Z \cup \{x_1, x_2\}$ or $Z \cup \{x_1, x_2, x_3\}$ is not a open geodetic dominatingset where $x_1, x_2, x_3 \notin Z$ and so $\gamma_{og}(G) \geq a$.

Now $S = Z \cup \{y, s, w, u\}$ is an open geodetic dominating set of $G$ so that $\gamma_{og}(G) = a$. Next we prove that $\gamma_{og}^+(G) = b$. Let $W = Z \cup \{v_1, v_2, ..., v_{b-a+1}, s, t, u\}$. Then $W$ is an open geodetic dominating set of $G$ and so $\gamma_{og}^+(G) \geq a - 4 + b - a + 1 + 3 = b$. First we prove that $W$ is a minimal open geodetic dominating set of $G$. Suppose that $W$ is not a minimal open geodetic dominating set of $G$. Then there exists $W' \subset W$ such that $W'$ is a open geodetic dominating set of $G$. Hence there exists $z \in W$ such that $z \notin W'$. By Theorem 1.1 (ii) $z \neq z_i (1 \leq i \leq a - 4)$. If $z = v_i (1 \leq i \leq b - a + 1)$ then $W'$ is not a dominating set of $G$. If $z = s$ or $t$ or $u$, then $W'$ is not an open geodetic set of $G$. Hence $W'$ is not an open geodetic dominating set of $G$. Therefore $W$ is a minimal open geodetic dominating set of $G$. Next we prove that $\gamma_{og}^+(G) = b$. Suppose that $\gamma_{og}^+(G) \geq b + 1$. Then there exists a open geodetic dominating set of $T$ such that $|T| \geq b + 1$. By Theorem 1.1 (ii) $Z \subset T$. Suppose that $v_i \notin T$ for some $i$. Then $z \notin T$ and either $v$ or $w \in T$. Let us assume that $v \in T$. Now $s$ and $v$ must lie on some pair of vertices of $T$.

Which implies $t$ must belongs to $T$. Hence $T$ contains open geodetic dominating set, which is a contradiction. Therefore $\gamma_{og}^+(G) = b$.

REFERENCES


