Multivalent Harmonic Function Associated With Salagean Operator

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ABSTRACT

In this paper we define, a class $HM(u,v,a)$ of m-valent harmonic functions involving Salagean Operator $D^m_\alpha$ is defined and studied. A subclass $THM(u,v,a)$ of a class $H(u,v,a)$ is also been defined and studied. integral operator, convolution condition, for functions belonging to subclass $THM(u,v,a)$ are obtained.

KEYWORDS: Multivalalent, Salagean, convolution, operator.

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1. INTRODUCTION

Definition 1.1

Let \( f \) be a harmonic function in a Jordan domain \( D \) with boundary \( C \). Suppose \( f \) is continuous in \( \overline{D} \) and \( f(z) \neq 0 \) on \( C \). Suppose \( f \) has no singular zeros in \( D \), and let \( m \) to be sum of the orders of the zeros of \( f \) in \( D \). Then \( \Delta_c \arg(f(z)) = 2\pi m \), where \( \Delta_c \arg(f(z)) \) denotes the change in argument of \( f(z) \) as \( z \) traverses \( C \).

It is also shown that if \( f \) is sense-preserving harmonic function near a point \( z_0 \), where \( f(z_0) = \omega_0 \) and if \( f(z) - \omega_0 \) has a zero of order \( m (m \geq 1) \) at \( z_0 \), then to each sufficiently small \( \varepsilon > 0 \) there corresponds a \( \delta > 0 \) with the property: “for each \( \alpha \in \mathbb{N}_\varepsilon(\omega_0) = \{ \omega : |\omega - \omega_0| < \delta \} \), the function \( f(z) - \alpha \) has exactly \( m \) zeros, counted according to multiplicity, in \( \mathbb{N}_\varepsilon(z_0) \)”. In particular, \( f \) has the open mapping property that is, it carries open sets to open sets.

Let \( \Delta \) be the open unit disc \( \Delta = \{ z : |z| < 1 \} \) also let \( a_k = b_k = 0 \) for \( 0 \leq k < m \) and \( a_m = 1 \). Ahuja and Jahangiri\(^5\)\(^9\) introduce and studied certain subclasses of the family \( \text{SH}(m) \), \( m \geq 1 \) of all multivalent harmonic and orientation preserving functions in \( \Delta \). A function \( f \) in \( \text{SH}(m) \) can be expressed as \( f = h + \overline{g} \), where \( h \) and \( g \) are of the form

\[
1.1 \quad h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1} \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1}, \quad |b_m| < 1.
\]

According to above argument, functions in \( \text{SH}(m) \) are harmonic and sense-preserving in \( \Delta \) if \( J_\tau > 0 \) in \( \Delta \). The class \( \text{SH}(1) \) of harmonic univalent functions was studied in details by Clunie and Sheil Small\(^{16}\). It was observed that \( m \)-valent mapping need not be orientation-preserving.
Let \( TH(m) \) denotes the subclass of \( SH(m) \) whose members are of the form

\[
h(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1}
\]

and

\[
g(z) = \sum_{n=1}^{\infty} |b_{n+m-1}| z^{n+m-1}, \quad |b_m| < 1.
\]

**Definition 1.2**

For analytic function \( h(z) \in S(m) \) Salagean \(^{33}\) introduced an operator \( D^\nu_m \) defined as follows:

\[
D^0_m h(z) = h(z), \quad D^1_m h(z) = D_m (h(z)) = \frac{Z}{m} h'(z) \quad \text{and}
\]

\[
D^\nu_m h(z) = D_m (D^{\nu-1}_m h(z)) = \frac{z(D^{\nu-1}_m h(z))'}{m}
\]

\[
= z + \sum_{n=2}^{\infty} \left( \frac{n + m - 1}{m} \right)^\nu a_{n+m-1} z^{n+m-1}, \quad \nu \in \mathbb{N}.
\]

Whereas, Jahangiri et al. \(^{17}\) defined the Salagean operator \( D^\nu_m f(z) \) for multivalent harmonic function as follows:

\[
D^\nu_m f(z) = D^\nu_m h(z) + (-1)^\nu D^\nu_m g(z)
\]

where,

\[
D^\nu_m h(z) = z^m + \sum_{n=2}^{\infty} \left( \frac{n + m - 1}{m} \right)^\nu a_{n+m-1} z^{n+m-1}
\]

\[
D^\nu_m g(z) = \sum_{n=1}^{\infty} \left( \frac{n + m - 1}{m} \right)^\nu b_{n+m-1} z^{n+m-1}.
\]
Now, a sub class $H_m(\lambda, v, \alpha)$ of $m$-valent harmonic functions involving Salagean operator $D_m^\nu f(z)$ is defined as follows:

**Definition 1.3**

Let $f(z) = h(z) + \overline{g(z)}$ be the harmonic multivalent function of the form (1.1), then $f$ belongs to $HM(u, v, a)$ if and only if

\[
\text{Re} \left\{ (1 - \lambda) \frac{D_m^\nu f(z)}{z^m} + \lambda \frac{\partial}{\partial \theta} z^m \right\} > \alpha
\]

where $0 \leq \alpha < 1, \lambda \geq 0, z = re^{i\theta} \in \Delta$ and $D_m^\nu f(z)$ is defined by (1.2) and

\[
\frac{\partial}{\partial \theta} D_m^\nu f(z) = i \left[ z(D_m^\nu h(z))^* - (-1)^r z(D_m^\nu g(z))^* \right], \quad \frac{\partial}{\partial \theta} z^m = imz^m.
\]

Denote the subclass $THM(u, v, a)$ consist of harmonic functions $f_v = h + g_v$ in $HM(u, v, a)$ so that $h$ and $g_v$ are of the form

\[
h(z) = \sum_{n=2}^{\infty} a_n z^{n-1},
\]

\[
g_v(z) = (-1)^r \sum_{n=1}^{\infty} b_n z^{n-1}, \quad \left| b_m \right| < 1.
\]

Also note that $THM(u, v, 0) = THM(u, v)$.

The class $HM(u, v, 0)$ provides a transition between two classes:

\[
\text{Re} \left\{ \frac{D_m^\nu f(z)}{z^m} \right\} > \alpha \quad \text{and} \quad \text{Re} \left\{ \frac{\partial}{\partial \theta} \frac{D_m^\nu f(z)}{z^m} \right\} > \alpha \quad \text{as} \quad \lambda \text{ moves between 0 and 1}.
\]

Denote $HM(0, v, a)$ by $PM(v, a)$ and $HM(1, v, a)$ by $QM(v, a)$. 
Definition 1.4

The generalized Bernardi-Libera-Livingston integral operator \( L_c(f(z)) \) for \( m \)-valent functions is defined by

\[
L_c(f(z)) = \frac{c + m}{z^c} \int_0^z t^{c-1}h(t)dt + \frac{c + m}{z^c} \int_0^z t^{c-1}g(t)dt, \quad c > -1.
\]

2. INTEGRAL OPERATOR

Let \( f \) belongs to \( \text{THM}(u,v,a) ; \quad \lambda \geq 1 \). Thus \( L_c(D^m f(z)) \) belongs to the class \( \text{THM}(u,v,a) \).

Proof

From the representation of \( L_c(f(z)) \) it follows that

\[
L_c(D^m f(z)) = \frac{c + m}{z^c} \int_0^z t^{c-1}D^m_h(t)dt + \frac{c + m}{z^c} \int_0^z t^{c-1}(-1)^\lambda D^m g(t)dt
\]

\[
= \frac{c + m}{z^c} \int_0^z t^{c-1} \left( t^m - \sum_{n=2}^{\infty} |a_{n+m-1}| t^{n+m-1} \right) dt
\]

\[
+ \frac{c + m}{z^c} (-1)^\lambda \int_0^z t^{c-1} \left( \sum_{n=2}^{\infty} |b_{n+m-1}| t^{n+m-1} \right) dt
\]

\[
= z^m - \sum_{n=2}^{\infty} A_{n+m-1} z^{n+m-1} + (-1)^\lambda \sum_{n=1}^{\infty} B_{n+m-1} z^{n+m-1}
\]

where, \( A_{n+m-1} = \frac{c + m}{n + m - 1 + c} |a_{n+m-1}| \), \( B_{n+m-1} = \frac{c + m}{n + m - 1 + c} |b_{n+m-1}| \)

Therefore,

\[
\sum_{n=2}^{\infty} \left( \frac{n+m-1}{m} \right)^\lambda \left[ \left( \frac{n+m-1}{m} \right)^\lambda + (1-\lambda) \right] \frac{c + m}{n + m - 1 + c} |a_{n+m-1}| +
\]
\[ + \left\{ \left( \frac{n + m - 1}{m} \right) \lambda - (1 - \lambda) \right\} \frac{c + m}{n + m - 1 + c} \left| b_{n+m-1} \right| \]

\[ \leq \sum_{n=2}^{\infty} \left\{ \left( \frac{n + m - 1}{m} \right)^\nu \left[ \left( \frac{n + m - 1}{m} \right) \lambda + (1 - \lambda) \right] \left| a_{n+m-1} \right| \right\} \]

\[ + \left\{ \left( \frac{n + m - 1}{m} \right)^\nu \left[ \left( \frac{n + m - 1}{m} \right) \lambda - (1 - \lambda) \right] \left| b_{n+m-1} \right| \right\} \]

\[ \leq (1 - \alpha) - (2\lambda - 1) \left| b_m \right| \]

and so the proof is complete.

3. CONVOLUTION PROPERTY

Let \( f_v \) belongs to \( THM(u,v,a) \) and \( F_v \) belongs to \( THM(u,v,a) \); \( \lambda \geq 1 \) then the convolution

\[
(f_v \ast F_v)(z) = z^m - \sum_{n=2}^{\infty} \left| a_{n+m-1}A_{n+m-1} \right| z^{n+m-1} + \]

\[ + (-1)^\nu \sum_{n=1}^{\infty} \left| b_{n+m-1}B_{n+m-1} \right| z^{n+m-1} \in TH_m(\lambda, v, \alpha). \]

Proof

For \( F_v \) belongs to \( THM(u,v,a) \) so, \( \left| A_{n+m-1} \right| \leq 1, \left| B_{n+m-1} \right| \leq 1. \)

Consider,

\[
\sum_{n=2}^{\infty} \left[ \left( \frac{n + m - 1}{m} \right)^\nu \left[ \left( \frac{n + m - 1}{m} \right) \lambda + (1 - \lambda) \right] \left| a_{n+m-1}A_{n+m-1} \right| \right] + \]

\[
\sum_{n=1}^{\infty} \left[ \left( \frac{n + m - 1}{m} \right)^\nu \left[ \left( \frac{n + m - 1}{m} \right) \lambda - (1 - \lambda) \right] \left| b_{n+m-1} \right| \right] \]

\[ \leq (1 - \alpha) - (2\lambda - 1) \left| b_m \right| \]
\[ \sum_{n=1}^{\infty} \left[ \frac{(n+m-1)^{\gamma} \left( \frac{n+m-1}{m} \lambda - (1-\lambda) \right) b_{n+m-1} B_{n+m-1}}{1-\alpha} \right] \]

\[ \leq \sum_{n=2}^{\infty} \left[ \frac{(n+m-1)^{\gamma} \left( \frac{n+m-1}{m} \lambda + (1-\lambda) \right) a_{n+m-1}}{1-\alpha} \right] + \]

\[ \sum_{n=1}^{\infty} \left[ \frac{(n+m-1)^{\gamma} \left( \frac{n+m-1}{m} \lambda - (1-\lambda) \right) b_{n+m-1}}{1-\alpha} \right] \]

\[ \leq 1 \text{ using equation coefficient inequality.} \]

Therefore the result follows.

REFERENCES


