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# Solution to the Slowly Rotating Polytropic Fluid Sphere by Ramanujan's Method

**Elias Priya A.*[1], Thomas Sanish[2] and Kumar Shashi[3]**

Department of Physics, FoS, Sam Higginbottom University of Agriculture, Technology & Sciences; Prayagraj-211007, [1]Email: eliaspriya702@gmail.com

## ABSTRACT

The polytropic models has been used for providing the approximate physical representations for a variety of classical astrophysical objects such as stars, planets and globular clusters which is a simplified model. Therefore, an attempt has been made to add rotation to the configuration and to obtain series for the slowly rotating polytropic fluid sphere using Special function. The new methodology i.e Ramanujan's method has been introduced in this paper, to obtain the boundary values of the non-linear differential equation associated with the structure of polytropic fluid sphere. It is the most efficient iterative method used to obtain the exact numerical solution of the given problem. Different parameters subjected to the polytropic index n has been calculated for n = 1.0, 1.5, 2.0 and 3.0. Further, the boundary values are compared with Chandrashekhar's results and it shows that Special function can be efficiently used to solve such types inhomogeneous second order differential equation.

**KEYWORDS*:*** Stellar Structure, nonlinear ODE, Lane Emden Equation, Series solution and Ramanujan method.

## *Corresponding Author

## Priya A. Elias

Department of Physics, FoS, Sam Higginbottom

University of Agriculture, Technology & Sciences; Prayagraj-211007

Email: eliaspriya702@gmail.com

## INTRODUCTION

Theoretical studies for the problem of the equilibrium structure of a rotating gaseous sphere are often carried out to understand the nature of internal structures responsible for various observed phenomena of the stars and it have been a very important investigation in the field of many astrophysicists. The rotation problems was first investigated by S. Chandrashekhar[1] using a first order perturbation, Daftar Dar & Jafari H.[3] proposed an iterative method for solving non-linear Volterra integral equations,Jabbar R.J.[5] integrated numerically the Chandrasekhar's equation for polytropic gas spheres of zero order and index $n = 0(0.1)4.9$ numerically with CDC7600 automatic computer to 16 decimal places,Kashem B.E & Shihab S.[6]used modified Hermite operation matrix method to solve the structure of slowly rotating polytropes, Mohan C. & Al-Bayaty A.R[7] proposed a power series method, Oproui T. &Horedt G.P.[8] used analytical method, Parks A.D[9] used Frobenius method & power series method, Prince A.M. & Thomas S.[10]introduced new iterative method to solve second ODE and used Ramanujan's method for series calculation, Elias P.A. & Thomas S.[11] solved the structure of slowly rotating polytropic fluid sphere by special function (Legendre and Power series method), Monaghan J.J & Roxburgh I.W[12] used an extension of Jean's generalized Roche model, Roxburgh I. & Stockman L.M.[13] used power series method for determining the structure of polytrope.

Stellar structure refers to the calculated properties of interior of a star which is a self-gravitational ball of gas held by its own gravity and completely gaseous all the way to the centre. A polytropic fluid sphere is a centrally condensed gaseous configuration governing hydrostatic equilibrium with centrifugal force and balancing gravity. The structure of polytropic fluid sphere consist of two parts:

- ***Inner Region:***

The configuration of inner most layer contains most of the material as it is centrally condensed. Its mass is much larger than the mass of the outer surface which is known as the approximation theory of stellar surfaces.

- ***Outer Region:***

In the case of outer region of rotating polytropic fluid sphere, rotation leads to the overall expansion. The density of the outer region is very small (negligible) due to larger centrifugal force.

The basic equations defining the structure of polytrope of index $n$, pressure $P$, density $\rho$, central density $\rho_c$ and constant $k$ such as,

$$P = k\rho^{1+\frac{1}{n}} (1)$$

$$\nabla P = \rho(\nabla\varphi + \Omega^2\widetilde{\omega}) (2)$$

$$\nabla^2\varphi = 4\pi G\rho \qquad (3)$$

Where, $\varphi$ is the gravitational potential satisfying the Poisson equation and $G$ is the gravitational constant.

Taking gradient of equation (2), we get

$$\vec{\nabla}\left[\frac{1}{\rho}\vec{\nabla}P\right] = -4\pi G\rho + 2\pi G\alpha\rho_c\widetilde{\omega} (4)$$

$$\rho = \rho_c\sigma^n,\ r = \alpha\xi,\ \alpha^2 = \left[\frac{(n+1)k\rho_c^{\frac{1}{n}-1}}{4\pi G}\right] (5)$$

Now, assuming the distance of a point on the surface from the origin r and the $\theta$ the angle between the rotation axis (z-axis) and $r$. We shall denote the cosine of the colatitudes $\theta$ by $\mu$.

$\mu$ =cosz (6)

Since, $\vec{\nabla}\left[\frac{1}{\rho}\vec{\nabla}P\right] = (n+1)K\lambda^{\frac{1}{n}}\nabla^2\sigma \qquad (7)$

Considering a parameter; $\alpha = \frac{\Omega^2}{2\pi G\rho_c}$, $\quad \left[\frac{(n+1)k\rho_c^{\frac{1}{n}-1}}{4\pi G}\right]\nabla^2\sigma = -\sigma^n + \frac{\Omega^2}{2\pi G\rho_c} (8)$

The above equation (8) becomes,

$$\nabla^2\sigma = -\sigma^n + \alpha (9)$$

On Solving the above equation, we have;

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\psi}{\partial\xi}\right) + \frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu}\right] = [-\sigma^n + \alpha]\psi \qquad (10)$$

## ***Boundary Condition*:**

At the centre the polytopes have maximum density by boundaryconditions, $\xi = 0,\ \theta = 1$ and $\theta' = 0 \qquad (11)$

and assuming that a solution of equation (10) to the last equation to first order in the small parameter using perturbation technique,

$\sigma = \theta + \alpha\psi + \alpha^2\varphi + \ldots \qquad (12)$

Expanding equation (9) in terms of equation (10) we have,

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left[\xi^2\left(\frac{\partial\theta}{\partial\xi}+\alpha\frac{\partial\psi}{\partial\xi}+\alpha^2\frac{\partial\varphi}{\partial\xi}+\cdots\right)\right]+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\left(\frac{\partial\theta}{\partial\mu}+\alpha\frac{\partial\psi}{\partial\mu}+\alpha^2\frac{\partial\varphi}{\partial\mu}+\cdots\right)\right]=$$

$$-\left(\theta^n+n\alpha\theta^{n-1}\psi+n\alpha^2\theta^{n-1}\varphi+\ldots+\frac{n(n-1)}{2}\alpha^2\theta^{n-2}\psi^2+\cdots\right)+\alpha \qquad (13)$$

Since, $\theta$ is a spherically-symmetrical function and independent of $\mu$.

Comparing the coefficients of zero, first and second order of $\alpha$, we obtain the following equation,

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\theta}{\partial\xi}\right)=-\theta^n \ (14)$$

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\psi}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu}\right]=-\,n\theta^{n-1}\psi+1 (15)$$

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\psi}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu}\right]=-\,n\theta^{n-1}\varphi-\frac{n(n-1)}{2}\theta^{n-2}\psi^2 (16)$$

Where, equation (14) is the Lane Emden equation of the first kind and equation (16) is the Lane Emden equation of the second kind for the slowly rotating polytropic fluid sphere of index n, which is the basic equation in the theory of stellar structure, in the sequence of paper[1-19]. To obtain the solution of equation (15) consider ψ as a series of Legendre Polynomials$P_m(\mu)$,

$$\psi=\sum_{m=0}^{\infty}A_m\Psi_mP_m(\mu)\text{and}\quad \varphi=\sum_{m=0}^{\infty}\varphi_mP_m(\mu)(17)$$

Where$A_m^{'}s$ are the arbitrary constants and $P_m(\mu)$.

The Legendre functions of index *m* satisfies the differential equation;

$$\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial P_m}{\partial\mu}\right)=-m(m+1)P_m(\mu)(18)$$

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\psi_0}{\partial\xi}\right)+\ n\theta^{n-1}\psi_0-1=0\ (19)$$

$$\left(\xi^2\frac{\partial^2\Psi_m}{\partial\xi^2}\right)+2\xi\left(\frac{\partial\Psi_m}{\partial\xi}\right)+[\xi^2-m(m+1)]\Psi_m(\xi)=0 \qquad (20)$$

$where\ \ m=1,2,3\ldots;\ and$ $\xi=0$ is regular singular point for 1. Consider its series solution,

$$\psi_m=a_0\xi^m+a_1\xi^{m+1}+a_2\xi^{m+2}+\ldots. \qquad (21)$$

Let $\psi_m=\sum_{r=0}^{\infty}A_r\xi^{k+r}$ be the solution,

$$\frac{\partial\psi_m}{\partial\xi}=\sum_{r=0}^{\infty}A_r(k+r)\xi^{k+r-1}\ (22)$$

$\frac{\partial^2 \psi_m}{\partial \xi^2} = \sum_{r=0}^{\infty} A_r (k+r)(k+r-1)\xi^{k+r-2}$(23)

Substituting this value in equation (20), we get;

$$\sum_{r=0}^{\infty} A_r (k+r)(k+r-1)\xi^{k+r-2} + 2\xi \sum_{r=0}^{\infty} A_r (k+r)\xi^{k+r-1} + [\xi^2 - m(m+1)] \sum_{r=0}^{\infty} A_r \xi^{k+r} = 0$$

The solution of equation (17) is then given by,

$$\psi_0 = 1 - \frac{1}{3!}\xi^2 + \frac{n}{5!}\xi^4 - \frac{n(8n-5)}{3{\cdot}7!}\xi^6 + \frac{n(70-183n+122n^2)}{9{\cdot}9!}\xi^8 + \ldots. \qquad (24)$$

$$\psi_2 = \xi^2 - \frac{1}{14}\xi^4 + \frac{n(10n-7)}{42{\cdot}36}\xi^6 - \frac{n(308n^2-503n+210)}{42{\cdot}36{\cdot}330}\xi^8 + \ldots.. \qquad (25)$$

which is the series solution of non-linear second order differential equation for of Lane-Emden equation near origin.

## RESULT & DISCUSSION

- The series solution for the slowly rotating polytropic fluid sphere equation,

$$\left(\xi^2 \frac{\partial^2 \Psi_m(\xi)}{\partial \xi^2}\right) + 2\xi\left(\frac{\partial \Psi_m(\xi)}{\partial \xi}\right) + [\xi^2 - m(m+1)]\Psi_m(\xi) = 0$$

  is evaluated for different polytropic index i.e.,$n = (0.0)1.0$, 1.5, 2.0 and 3.0. by using special function (Legendre function and power series).
- The boundary values $\xi = 0$, $\theta = 1$ *and* $\theta' = 0$ forsmall values of $\xi$ has been obtained using Ramanujan's Method and it is compared with Chandrashekhar's results (reference), as presented in the table. The percentage error shows the efficiency of solving such equations by Ramanujan's Method.

- ***Evaluation Table:***

**Table (1):Comparison table for the boundary values $\psi_0(\xi_1)$ using Ramanujan's Method with Chandrashekhar's Value.**

| n | Chandrashekhar's result $\psi_0(\xi_1)$ | Author's result $\psi_0(\xi_1)$ | % error |
|---|---|---|---|
| 1.0 | 3.1415 | 3.1414 | 0.00286 |
| 1.5 | 3.6538 | 3.6331 | 0.56810 |
| 2.0 | 4.3529 | 4.5825 | 5.01181 |
| 3.0 | 6.8968 | 6.4807 | 6.42060 |

**Graph (1): Comparison between the boundary values of Chandrashekhar's Result and Author's Result for Ψ0**

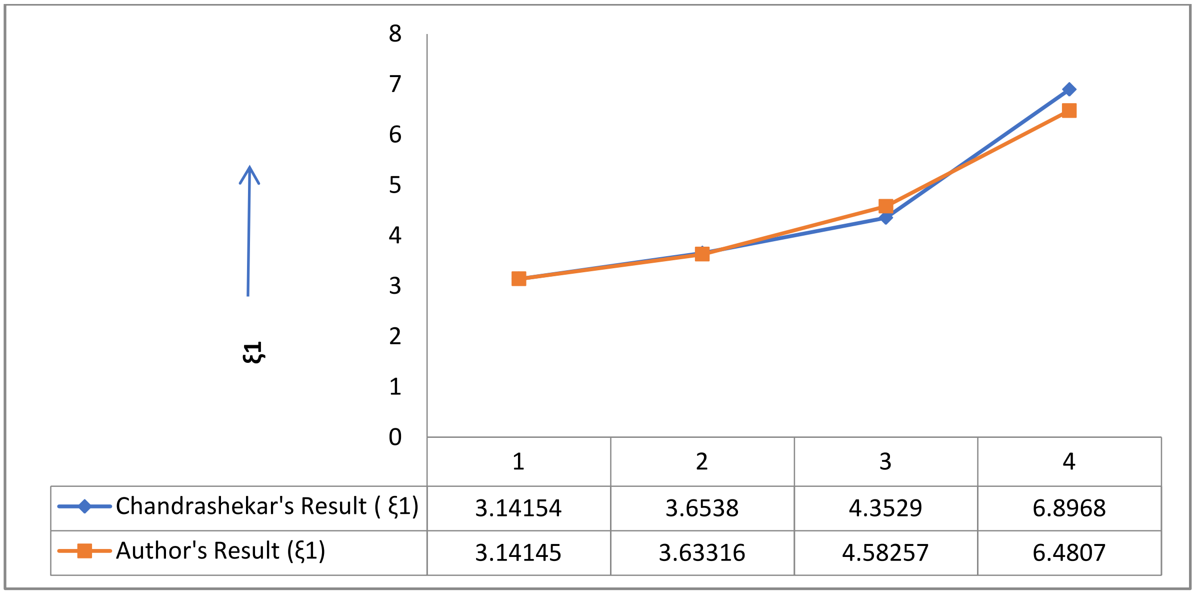


**Table (2): Comparison table for the boundary values $\psi_2(\xi_1)$ using Ramanujan's Method with Chandrashekhar's Value**

| n | Chandrashekhar's result $\psi_2(\xi_1)$ | Author's result $\psi_2(\xi_1)$ | % error |
|---|---|---|---|
| 1.0 | 4.5594 | 4.78634 | 4.74140 |
| 1.5 | 4.7820 | 4.72256 | 1.25863 |
| 2.0 | 5.6431 | 6.66862 | 15.3782 |
| 3.0 | 11.278 | 9.67048 | 16.6229 |

**Graph (2): Comparison between the boundary values of Chandrashekhar's Result and Author's Result for Ψ2**


## SUMMARY & CONCLUSION

In this paper, we have calculated the roots for the slowly rotating polytropic fluid spheres. The general Lane- Emden's equation has been converted to a special function. The series has been obtained using special function isfurther,solved by using a novel methodi.e., Ramanujan's method which can be seen as a useful tool for solving the inhomogeneous second order differential equation.

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